

GRID FUNCTIONS OF NONSTANDARD ANALYSIS IN THE THEORY OF DISTRIBUTIONS AND IN PARTIAL DIFFERENTIAL EQUATIONS

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The theory of distributions, pioneered by Dirac in [28] and developed in the first half of the XX Century, has become one of the fundamental tools of functional analysis. In particular, the possibility to define the weak derivative of a non-differentiable function has allowed the formulation and the study of a wide variety of nonsmooth phenomena by the theory of partial differential equations. However, the lack of a nonlinear theory of distributions is a limiting factor both for the applications and for the theoretical study of nonlinear PDEs. On the one hand, in the description of some physical phenomena such as shock waves and relativistic fields, it arises the need to have some mathematical objects which cannot be formalized in the sense of

distributions (we refer to [18] for some examples). On the other hand, the absence of a nonlinear theory of distributions poses some limitations in the study of nonlinear partial differential equations: while some nonlinear problems can be solved by studying the limit of suitable regularized problems, other problems do not allow for solutions in the sense of distributions (see for instance the discussion in [30]).

In 1954, L. Schwartz proved that the absence of a nonlinear theory for distributions is intrinsic: more formally, the main theorem of [49] entails that there is no differential algebra $(A, +, \otimes, D)$ in which the real distributions \mathcal{D}' can be embedded and the following conditions are satisfied:

- (1) \otimes extends the product over C^0 functions;
- (2) D extends the distributional derivative ∂ ;
- (3) the product rule holds: $D(u \otimes v) = (Du) \otimes v + u \otimes (Dv)$.

Despite this negative result, there have been many attempts at defining some notions of product between distributions (see for instance [19, 36]). Following this line of research, Colombeau in 1983 proposed an organic approach to a theory of generalized functions [17]: Colombeau's idea is to embed the distributions in a differential algebra with a good nonlinear theory, but at the cost of sacrificing the coherence between the product of the differential algebra with the product over C^0 functions. This approach has been met with interest and has proved to be a prolific field of research. For a survey of the approach by Colombeau and for recent advances, we refer to [18].

Research about generalized functions beyond distributions is also being carried out within the setting of nonstandard analysis. Possibly the earliest result in this sense is the proof by Robinson that the distributions can be represented by smooth functions of nonstandard analysis and by polynomials of a hyperfinite degree [45]. Distributions have also been represented by functions defined on hyperfinite domains, for instance by Kinoshita in [35] and, with a different approach, by Sousa Pinto and Hoskins in [33]. Another nonstandard approach to the theory of generalized functions has been proposed by Oberbuggenberg and Todorov in [43] and further studied by Todorov et al. [56, 57]. In this approach, the distributions are embedded in an algebra of asymptotic functions defined over a Robinson field of asymptotic numbers. Moreover, this algebra of asymptotic functions can be seen as a generalized Colombeau algebra where the set of scalars is an algebraically closed field rather than a ring with zero divisors. In this setting, it is possible to study generalized solutions to differential equations, and in particular to those with nonsmooth coefficient and distributional initial data [26, 41].

Another theory of generalized functions oriented towards the applications in the field of partial differential equations and of the calculus of variations has been developed by Benci and Luperi Baglini. In [4] and subsequent papers [5, 6, 7, 8], the authors developed a theory of ultrafunctions, i.e.

nonstandard vector spaces of a hyperfinite dimension that extend the space of distributions. In particular, the space of distributions can be embedded in an algebra of ultrafunctions V such that the following inclusions hold: $\mathcal{D}'(\mathbb{R}) \subset V \subset {}^*C^1(\mathbb{R})$ [7]. This can be seen as a variation on a result by Robinson and Bernstein, that in [11] showed that any Hilbert space H can be embedded in a hyperfinite dimensional subspace of *H . In the setting of ultrafunctions, some partial differential equations can be formulated coherently by a Galerkin approximation, while the problem of finding the minimum of a functional can be turned to a minimization problem over a formally finite vector space. For a discussion of the applications of ultrafunctions to functional analysis, we refer to [4, 6, 8].

The idea of studying the solutions to a partial differential equation via a hyperfinite Galerkin approximation is not new. For instance, Capiński and Cutland in [16] studied statistical solutions to parabolic differential equations by discretizing the equation in space by a Galerkin approximation in an hyperfinite dimension. The nonstandard model becomes then a hyperfinite system of ODEs that, by transfer, has a unique nonstandard solution. From this solution, the authors showed that it is possible to define a standard weak solution to the original problem. In the subsequent [13], the authors proved the existence of weak and statistical solutions to the Navier-Stokes equations in 3-dimensions by modelling the equations with a similar hyperfinite Galerkin discretization in space. This approach has spanned a whole line of research on the Navier-Stokes equations, concerning both the proof of the existence of solutions (see for instance [15, 22]) and the definition and the existence of attractors (see for instance [14, 23]). One of the advantages of this approach is that, by a hyperfinite discretization in space, the nonstandard models have a unique global solution, even when the original problem does not. For a discussion of the relation between the uniqueness of the solutions of the nonstandard formulation and the non-uniqueness of the weak solutions of the original problem in the case of the Navier-Stokes equations, we refer to [13].

In the theoretical study of nonlinear partial differential equations, sometimes problems do not allow even for a weak solution. However, the development of the notion of Young measures, originally introduced by L. C. Young in the field of optimal control in [61], has allowed for a synthetic characterization of the behaviour of the weak- \star limit of the composition between a nonlinear continuous function and a uniformly bounded sequence in L^∞ . By enlarging the class of admissible solutions to include Young measures, one can define generalized solutions for some class of nonlinear problems as the weak- \star limit of the solutions to a sequence of regularized problems [27, 30, 39, 40, 44, 50, 51]. A similar approach can be carried out in the field of optimal controls, where generalized controls in the sense of Young measures can be defined as the measure-valued limit points of a minimizing

sequence of controls. For an in-depth discussion of the role of Young measures as generalized solution to PDEs and as generalized controls, we refer to [2, 30, 53, 59].

In [20, 21, 24], Cutland showed that Young measures can be interpreted also as the standard part of internal controls of nonstandard analysis. The possibility to obtain a Young measure from a nonstandard control allows to study generalized solutions to nonlinear variational problems by means of nonstandard techniques: such an approach has been carried out for instance by Cutland in the aforementioned papers, and by Tuckey in [58]. For a discussion of this field of research, we refer to [42].

Structure of the paper. In this paper, we will discuss another theory of generalized functions of nonstandard analysis, hereafter called grid functions (see Definition 1.1), that provide a coherent generalization both of the space of distributions and of a space of parametrized measures that extends the space of Young measures. In Section 1, we will define the space of grid functions, and recall some well-established nonstandard results that will be used throughout the paper. In particular, we will formulate in the setting of grid functions some known results regarding the relations between the hyperfinite sum and the Riemann integral, and the finite difference operators of an infinitesimal step and the derivative of a C^1 function.

In Section 2, we will study the relations between the grid functions and the distributions, with the aim of proving that every distribution can be obtained from a suitable grid function. In order to reach this result, we will introduce an algebra of nonstandard test functions that can be seen as the grid function counterpart to the space $\mathcal{D}(\Omega)$ of smooth functions with compact support over $\Omega \subseteq \mathbb{R}^k$. By duality with respect to the algebra of test functions, we will define a module of bounded grid functions, and an equivalence relation between grid functions (see Definition 2.3 and Definition 2.5). We will then prove that the set of equivalence classes of bounded grid functions with respect to this equivalence relation is a real vector space that is isomorphic to the space of distributions. Afterwards, we will discuss how the finite difference operators generalize not only the usual derivative for C^1 functions, but also the distributional derivative.

After having shown that the finite difference operator generalizes the distributional derivative, our study of the relations between grid functions and distributions concludes with a discussion of the Schwartz impossibility theorem. In particular, we will show that the space of distributions can be embedded in the space of grid functions in a way that

- (1) the product over the grid functions generalizes the pointwise product between continuous functions;
- (2) the finite difference is coherent with the distributional derivative modulo the equivalence relation induced by duality with test functions;
- (3) a discrete chain rule for products holds.

This theorem supports our claim that the space of grid functions provides a nontrivial generalization of the space of distributions.

In Section 3, we will embed the space of grid functions in the spaces ${}^*L^p$ with $1 \leq p \leq \infty$, and we will study some properties of grid functions through this embedding. Moreover, we will discuss a generalization of the embedding of $L^2(\Omega)$ in a hyperfinite subspace of ${}^*L^2(\Omega)$ due to Robinson and Bernstein [11]. This classic result will be generalized in two directions:

- (1) for every $1 \leq p \leq \infty$, we will embed the spaces $L^p(\Omega)$ in the space of grid functions, which is a subspace of ${}^*L^p(\Omega)$ of a hyperfinite dimension;
- (2) the above embedding is actually an embedding of the bigger space $\mathcal{D}'(\Omega)$ into a hyperfinite subspace of ${}^*L^p(\Omega)$ for all $1 \leq p \leq \infty$.

Moreover, this embedding is obtained with different techniques from the original result by Robinson and Bernstein.

In the second part of Section 3, we will establish a correspondence between grid functions and parametrized measures, in a way that is coherent with the isomorphism between equivalence classes of bounded grid functions and distributions discussed in Section 2. The results discussed in Section 3 will be used in Section 4, where we will discuss the grid function formulation of partial differential equations, in Section 5, where we will show selected applications of grid functions from different fields of functional analysis, and in the paper [12], where we will study in detail a grid function formulation of a class of ill-posed partial differential equations with variable parabolicity direction.

In Section 4, we will discuss how to formulate partial differential equations in the space of grid functions in a way that coherently generalizes the standard notions of solutions. In particular, stationary PDEs will be given a fully discrete formulation, while time-dependent PDEs will be given a continuous-in-time and discrete-in-space formulation, resulting in a hyperfinite system of ordinary differential equations, as in the nonstandard formulation of the Navier-Stokes equations by Capiński and Cutland.

In Section 5, we will use the theory of grid functions developed so far to study two problems in the nonlinear theory of distributions and in the calculus of variations. These problems are classically studied within different frameworks, but we will show that each of these problems can be formulated in the space of grid functions in a way that the nonstandard solutions generalize the respective standard solutions.

1. TERMINOLOGY AND PRELIMINARY NOTIONS

In this section, we will now fix some notation and recall some results from nonstandard analysis that will be useful throughout the paper.

If $A \subseteq \mathbb{R}^k$, then \overline{A} is the closure of A with respect to any norm in \mathbb{R}^k , ∂A is the boundary of A , and χ_A is the characteristic function of A . If $x \in \mathbb{R}$, then $\chi_x = \chi_{\{x\}}$. If $f : A \rightarrow \mathbb{R}$, $\text{supp } f$ is the closure of the set

$\{x \in A : f(x) \neq 0\}$. These definitions are generalized as expected also to nonstandard objects.

We consider the following norms over \mathbb{R}^k and ${}^*\mathbb{R}^k$: if $x \in \mathbb{R}^k$ or $x \in {}^*\mathbb{R}^k$, then $|x| = \sqrt{\sum_{i=1}^k x_i^2}$ is the euclidean norm, and $|x|_\infty = \max_{i=1, \dots, k} |x_i|$ is the maximum norm.

We will denote by ${}^*\mathbb{R}_{fin}$ the set of finite numbers in ${}^*\mathbb{R}$, i.e. ${}^*\mathbb{R}_{fin} = \{x \in {}^*\mathbb{R} : x \text{ is finite}\}$. The notion of finiteness can be extended componentwise to elements of ${}^*\mathbb{R}^k$ whenever $k \in \mathbb{N}$: we will say that $x \in {}^*\mathbb{R}^k$ is finite iff all of its components are finite, and we define ${}^\circ x = ({}^\circ x_1, {}^\circ x_2, \dots, {}^\circ x_k) \in \mathbb{R}^k$. Similarly, if $x, y \in {}^*\mathbb{R}^k$, we will write $x \approx y$ if $|x - y| \approx 0$ (notice that this is equivalent to $|x - y|_\infty \approx 0$).

We will denote by e_1, \dots, e_k the canonical basis of \mathbb{R}^k and of ${}^*\mathbb{R}^k$. If $f : A \subseteq {}^*\mathbb{R}^m \rightarrow {}^*\mathbb{R}^k$, we will denote by f_1, \dots, f_k the hyperreal valued functions that satisfy the equality $f(x) = (f_1(x), \dots, f_k(x))$ for all $x \in {}^*\mathbb{R}$.

Let $\Omega \subseteq \mathbb{R}^k$ be an open set or the closure of an open set. We will often reference the following real vector spaces:

- $C_b^0(\Omega) = \{f \in C^0(\Omega) : f \text{ is bounded and } \lim_{|x| \rightarrow \infty} f(x) = 0\}$.
- $C_c^0(\Omega) = \{f \in C_b^0(\Omega) : \text{supp } f \subset\subset \Omega\}$.
- $\mathcal{D}(\Omega) = \{f \in C^\infty(\Omega) : \text{supp } f \subset\subset \Omega\}$.
- In the sequel, measurable will mean measurable with respect to μ_L , the Lebesgue measure over \mathbb{R}^n . Consider the equivalence relation given by equality almost everywhere: two measurable functions f and g are equivalent if $\mu_L(\{x \in \Omega : f(x) \neq g(x)\}) = 0$. We will not distinguish between the function f and its equivalence class, and we will say that $f = g$ whenever the functions f and g are equal almost everywhere.

For all $1 \leq p < \infty$, $L^p(\Omega)$ is the set of equivalence classes of measurable functions $f : \Omega \rightarrow \mathbb{R}$ that satisfy

$$\int_{\Omega} |f|^p dx < \infty.$$

If $f \in L^p(\Omega)$, the L^p norm of f is defined by

$$\|f\|_p^p = \int_{\Omega} |f|^p dx.$$

$L^\infty(\Omega)$ is the set of equivalence classes of measurable functions that are essentially bounded: we will say that $f : \Omega \rightarrow \mathbb{R}$ belongs to $L^\infty(\Omega)$ if there exists $y \in \mathbb{R}$ such that $\mu_L(\{x \in \Omega : |f(x)| > y\}) = 0$. In this case,

$$\|f\|_\infty = \inf\{y \in \mathbb{R} : \mu_L(\{x \in \Omega : |f(x)| > y\}) = 0\}.$$

If $1 < p < \infty$, we recall that p' is defined as the unique solution to the equation

$$\frac{1}{p} + \frac{1}{p'} = 1,$$

while $1' = \infty$ and $\infty' = 1$.

- $\mathbb{M}(\mathbb{R}) = \{\nu : \nu \text{ is a Radon measure over } \mathbb{R} \text{ satisfying } |\nu|(\mathbb{R}) < +\infty\}$.
- $\mathbb{M}^{\mathbb{P}}(\mathbb{R}) = \{\nu \in \mathbb{M}(\mathbb{R}) : \nu \text{ is a probability measure}\}$.

Following [2, 3, 59] and others, measurable functions $\nu : \Omega \rightarrow \mathbb{M}^{\mathbb{P}}(\mathbb{R})$ will be called Young measures. Measurable functions $\nu : \Omega \rightarrow \mathbb{M}(\mathbb{R})$ will be called parametrized measures, even though in the literature the term parametrized measure is used as a synonym for Young measure. If ν is a parametrized measure and if $x \in \Omega$, we will write ν_x instead of $\nu(x)$.

If $f \in C^1(\mathbb{R})$, we will denote the derivative of f by $\frac{df}{dx}$, f' or Df . If $f : [0, T] \times \Omega \rightarrow \mathbb{R}$, we will think of the first variable of f as the time variable, denoted by t , and we will write f_t for the derivative $\frac{\partial f}{\partial t}$. We adopt the multi-index notation for partial derivatives and, if α is a multi-index, we will denote by $D^\alpha f$ the function

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_k^{\alpha_k}}.$$

If $\alpha = (\alpha_1, \dots, \alpha_k)$ is a multi-index, then $\alpha - e_i = (\alpha_1, \dots, \alpha_i - 1, \dots, \alpha_k)$.

We recall that a real distribution over Ω is an element of $\mathcal{D}'(\Omega)$, i.e. a continuous linear functional $T : \mathcal{D}(\Omega) \rightarrow \mathbb{R}$. If T is a distribution and φ is a test function, we will denote the action of T over φ by $\langle T, \varphi \rangle_{\mathcal{D}(\Omega)}$. When T can be identified with a L^p function, we will sometimes write $\int_{\Omega} T \varphi dx$ instead of $\langle T, \varphi \rangle_{\mathcal{D}(\Omega)}$.

If $T \in \mathcal{D}'(\mathbb{R})$, we will denote the derivative of T by T' or DT . Recall that T' is defined by the formula

$$\langle DT, \varphi \rangle_{\mathcal{D}(\Omega)} = -\langle T, D\varphi \rangle_{\mathcal{D}(\Omega)}.$$

If $T \in \mathcal{D}'(\Omega)$ and α is a multi-index, the distribution $D^\alpha T$ is defined in a similar way:

$$\langle D^\alpha T, \varphi \rangle_{\mathcal{D}(\Omega)} = (-1)^{|\alpha|} \langle T, D^\alpha \varphi \rangle_{\mathcal{D}(\Omega)}.$$

It is well-established that the distributional derivative allows to define a notion of weak derivative for L^p functions (see for instance [52, 54]). L^2 functions whose weak derivatives up to order $p < \infty$ are still L^2 functions are of a particular relevance in the study of partial differential equations. We will now recall the definition of the space of such functions. For $p \in \mathbb{N}$, $p \geq 1$, the space $H^p(\Omega)$ is defined as

$$H^p(\Omega) = \{f \in L^2(\Omega) : D^\alpha f \in L^2(\Omega) \text{ for every multi-index } \alpha \text{ with } |\alpha| \leq p\}.$$

We also consider the following norm over the space $H^p(\Omega)$:

$$\|f\|_{H^p} = \sum_{|\alpha| \leq p} \|D^\alpha f\|_2,$$

and we will call it the H^p norm. Recall also that $H_0^p(\Omega) \subset H^p(\Omega)$ is defined as the closure of $\mathcal{D}(\Omega)$ in $H^p(\Omega)$ with respect to the H^p norm. For the properties of the space $H^p(\Omega)$ and of the space $H_0^p(\Omega)$, we refer to [52, 54].

We will now introduce the space of grid functions.

Definition 1.1. Let $N_0 \in {}^*\mathbb{N}$ be an infinite hypernatural number. Set $N = N_0!$ and $\varepsilon = 1/N$, and define

$$\mathbb{X} = \{n\varepsilon : n \in [-N^2, N^2] \cap {}^*\mathbb{Z}\}.$$

We will say that an internal function $f : \mathbb{X}^k \rightarrow {}^*\mathbb{R}$ is a grid function and, if $A \subseteq \mathbb{X}^k$ is internal, we denote by ${}^*\mathbb{R}^A$ the space of grid functions defined over A : ${}^*\mathbb{R}^A = \text{Intl}({}^*\mathbb{R}^A) = \{f : A \rightarrow {}^*\mathbb{R} \text{ and } f \text{ is internal}\}.$

1.1. Some elements of nonstandard topology. In the next definition, we will give a canonical extension of subsets of the standard euclidean space \mathbb{R}^k to internal subsets of the grid \mathbb{X}^k .

Definition 1.2. For any $A \subseteq \mathbb{R}^k$, we define $A_{\mathbb{X}} = {}^*A \cap \mathbb{X}^k$. Notice that $A_{\mathbb{X}}$ is an internal subset of \mathbb{X}^k , and in particular it is hyperfinite.

In general, we expect that for a generic set $A \subseteq {}^*\mathbb{R}^k$, ${}^\circ A_{\mathbb{X}} \neq \overline{A}$. For instance, if $A \cap \mathbb{Q}^k = \emptyset$, then $A_{\mathbb{X}} = {}^\circ A_{\mathbb{X}} = \emptyset$. In this section, we will prove that if A is an open set, then indeed $A_{\mathbb{X}}$ is a faithful extension of A , in the sense that ${}^\circ A_{\mathbb{X}} = {}^\circ \overline{A_{\mathbb{X}}} = \overline{A}$. Moreover, there is a nice characterization of the boundary of $A_{\mathbb{X}}$ which is projected to the boundary of A via the standard part map.

In order to prove these results, we need to show that for an open set A , $\mu(x) \cap {}^*A \neq \emptyset$ is equivalent to $\mu(x) \cap A_{\mathbb{X}} \neq \emptyset$ for all $x \in \overline{A}$.

Lemma 1.3. If $A \subseteq \mathbb{R}^k$ is an open set, then for all $x \in \overline{A}$ it holds

$$(1) \quad \mu(x) \cap {}^*A \neq \emptyset \iff \mu(x) \cap A_{\mathbb{X}} \neq \emptyset.$$

Proof. Let $x \in \overline{A}$. The hypothesis $N = N_0!$ for an infinite $N_0 \in {}^*\mathbb{N}$ ensures that for all $p \in \mathbb{Q}^k$, $p \in \mathbb{X}^k$. As a consequence, for all $n \in \mathbb{N}$ there exists $p \in A_{\mathbb{X}}$ with $|x - p| < 1/n$. By overspill, for some infinite $M \in {}^*\mathbb{N}$ there exists $p \in A_{\mathbb{X}}$ that satisfies $|x - p| < 1/M$. \square

We want to define a boundary for the set $A_{\mathbb{X}}$ that is coherent with the usual notion of boundary for A . The idea is to define the \mathbb{X} -boundary of $A_{\mathbb{X}}$ as the set of points of $A_{\mathbb{X}}$ that are within a step of length ε from a point of ${}^*A^c$.

Definition 1.4. Let $A \subseteq \mathbb{X}^k$. We define the \mathbb{X} -boundary of A as

$$\partial_{\mathbb{X}} A = \{x \in A : \exists y \in {}^*A^c \text{ satisfying } |x - y|_{\infty} \leq \varepsilon\}.$$

This definition is coherent with the usual boundary of an open set.

Proposition 1.5. Let $A \subseteq \mathbb{R}^k$ be an open set. Then ${}^\circ A_{\mathbb{X}} = \overline{A}$ and ${}^\circ(\partial_{\mathbb{X}} A_{\mathbb{X}}) = \partial A$.

Proof. The equality ${}^\circ A_{\mathbb{X}} = \overline{A}$ is a consequence of Lemma 1.3.

Recall the nonstandard characterization of the boundary of A : $x \in \partial A$ if and only if there exists $y \in {}^*A$, $x \neq y$, and $z \in {}^*A^c$ with $x \approx y \approx z$. This is sufficient to conclude that $\partial A \supseteq {}^\circ(\partial_{\mathbb{X}} A_{\mathbb{X}})$.

To prove that the other inclusion holds, we only need to show that if $x \in \partial A$, then there exists $y \in \partial_{\mathbb{X}} A_{\mathbb{X}}$ with $y \approx x$. Let $x \in \partial A$: since $A_{\mathbb{X}}$ is a hyperfinite set, we can pick $y \in A_{\mathbb{X}}$ satisfying

$$|{}^*x - y|_{\infty} = \min_{z \in A_{\mathbb{X}}} \{|{}^*x - z|_{\infty}\}.$$

Recall that $x \in A^c$, since A is open: as a consequence, for our choice of y we have $y \neq {}^*x$ and $|{}^*x - y|_{\infty} > 0$. We claim that $y \in \partial_{\mathbb{X}} A_{\mathbb{X}}$. In fact, suppose towards a contradiction that $y \notin \partial_{\mathbb{X}} A_{\mathbb{X}}$: in this case, for all $z \in A^c$, $|y - z|_{\infty} > \varepsilon$ and, in particular, $|{}^*x - y|_{\infty} > \varepsilon$. Let ${}^*x - y = \sum_{i=1}^k a_i e_i$, let $I = \{i \leq k : |a_i| = |{}^*x - y|_{\infty}\}$, and define

$$\tilde{y} = y + \sum_{i \in I} \frac{a_i}{|a_i|} \varepsilon e_i.$$

Since $|\tilde{y} - y|_{\infty} = \varepsilon$ and since $y \notin \partial_{\mathbb{X}} A_{\mathbb{X}}$, then $\tilde{y} \in A_{\mathbb{X}}$. Moreover,

$$|{}^*x - \tilde{y}|_{\infty} = \max_{i \notin I} \{|{}^*x - y| - \varepsilon, |a_i|\} < |{}^*x - y|_{\infty},$$

contradicting $|{}^*x - y|_{\infty} = \min_{z \in A_{\mathbb{X}}} \{|{}^*x - z|_{\infty}\}$. \square

From now on, let $\Omega \subseteq \mathbb{R}^k$ be an open set or the closure of an open set. By Proposition 1.5, this hypothesis is sufficient to ensure the equalities ${}^{\circ}\Omega_{\mathbb{X}} = \overline{\Omega}$ and ${}^{\circ}(\partial_{\mathbb{X}} \Omega_{\mathbb{X}}) = \partial \Omega$.

1.2. Derivatives and integrals of grid functions. Since grid functions are defined on a discrete set, there is no notion of derivative for grid functions. However, in nonstandard analysis it is fairly usual to replace the derivative by a finite difference operator with an infinitesimal step.

Definition 1.6 (Grid derivative). *For an internal grid function $f \in {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$, we define the i -th forward finite difference of step ε as*

$$\mathbb{D}_i f(x) = \mathbb{D}_i^+ f(x) = \frac{f(x + \varepsilon e_i) - f(x)}{\varepsilon}$$

and the i -th backward finite difference of step ε as

$$\mathbb{D}_i^- f(x) = \frac{f(x) - f(x - \varepsilon e_i)}{\varepsilon}.$$

If $n \in {}^\mathbb{N}$, \mathbb{D}_i^n is recursively defined as $\mathbb{D}_i(\mathbb{D}_i^{n-1})$ and, if α is a multi-index, then \mathbb{D}^{α} is defined as expected:*

$$\mathbb{D}^{\alpha} f = \mathbb{D}_1^{\alpha_1} \mathbb{D}_2^{\alpha_2} \dots \mathbb{D}_n^{\alpha_n} f.$$

These definitions can be extended to \mathbb{D}^- by replacing every occurrence of \mathbb{D} with \mathbb{D}^- .

For further details about the properties of the finite difference operators, we refer to Hanqiao, St. Mary and Wattenberg [32], to Keisler [34] and to van den Berg [9, 10].

Remark 1.7. Notice that if $f \in {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$ and if α is a standard multi-index, then $\mathbb{D}^\alpha f$ is not defined on all of $\Omega_{\mathbb{X}}$. However, if we let

$$\Omega_{\mathbb{X}}^\alpha = \{x \in \Omega_{\mathbb{X}} : \mathbb{D}^\alpha f \text{ is defined}\} = \{x \in \Omega_{\mathbb{X}} : x + \alpha\varepsilon \in \Omega_{\mathbb{X}}\}$$

then we have ${}^\circ\Omega_{\mathbb{X}}^\alpha = {}^\circ\Omega_{\mathbb{X}} = \overline{\Omega}$, since for every $x \in \Omega_{\mathbb{X}}^\alpha$ we have $x + \alpha\varepsilon \in \Omega_{\mathbb{X}}$ and $x + \alpha\varepsilon \approx x$ by the standardness of α .

In a similar way, if we define

$$\partial_{\mathbb{X}}^\alpha \Omega_{\mathbb{X}} = \{x \in \Omega_{\mathbb{X}} : x + \alpha\varepsilon \in \partial_{\mathbb{X}} \Omega_{\mathbb{X}}\},$$

then, from the relation $x + \alpha\varepsilon \approx x$ and from Proposition 1.5, we deduce that it holds also the equality ${}^\circ\partial_{\mathbb{X}}^\alpha \Omega_{\mathbb{X}} = {}^\circ\partial_{\mathbb{X}} \Omega_{\mathbb{X}} = \partial\Omega$. In section 4.1, we will use this result in order to show how Dirichlet boundary conditions can be expressed in the sense of grid functions.

Since $\Omega_{\mathbb{X}}^\alpha$ is a faithful extension of Ω in the sense of proposition 1.5, we will often abuse notation and write $\mathbb{D}^\alpha f \in {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$ instead of the correct $\mathbb{D}^\alpha f \in {}^*\mathbb{R}^{\Omega_{\mathbb{X}}^\alpha}$.

In the setting of grid functions, integrals are replaced by hyperfinite sums.

Definition 1.8 (Grid integral and inner product). Let $f, g : {}^*\Omega \rightarrow {}^*\mathbb{R}$ and let $A \subseteq \Omega_{\mathbb{X}} \subseteq \mathbb{X}^k$ be an internal set. We define

$$\int_A f(x) d\mathbb{X}^k = \varepsilon^k \cdot \sum_{x \in A} f(x)$$

and

$$\langle f, g \rangle = \int_{\mathbb{X}^k} f(x)g(x) d\mathbb{X}^k = \varepsilon^k \cdot \sum_{x \in \mathbb{X}^k} f(x)g(x),$$

with the convention that, if $x \notin {}^*\Omega$, $f(x) = g(x) = 0$.

A simple calculation shows that the fundamental theorem of calculus holds. In particular, for all $f : {}^*\mathbb{R}^{\mathbb{X}} \rightarrow {}^*\mathbb{R}$ and for all $a, b \in \mathbb{X}$, $b < N$, we have

$$\varepsilon \sum_{x=a}^b \mathbb{D}f(x) = f(b + \varepsilon) - f(a) \text{ and } \mathbb{D} \left(\varepsilon \sum_{x=a}^b f(x) \right) = f(b + \varepsilon).$$

The next Lemma is a well-known compatibility result between the grid integral and the Riemann integral of continuous functions.

Lemma 1.9. Let $\Omega \subset \mathbb{R}^k$ be a compact set. If $f \in C^0(\Omega)$, then

$$\int_{\Omega_{\mathbb{X}}} {}^*f(x) d\mathbb{X}^k \approx \int_{\Omega} f(x) dx.$$

Proof. See for instance Section 1.11 of [38]. □

1.3. S^α functions and C^α functions. We will now recall some well-known facts about S -continuity. This property has been widely used as a bridge between discrete functions of nonstandard analysis and standard continuous functions.

Definition 1.10. We will say that $x \in \Omega_{\mathbb{X}}$ is nearstandard in Ω iff there exists $y \in \Omega$ such that $x \approx y$.

Definition 1.11. We say that a function $f \in {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$ is S -continuous on $\Omega_{\mathbb{X}}$ iff $f(x)$ is finite for some nearstandard $x \in \Omega_{\mathbb{X}}$ and for every nearstandard $x, y \in \Omega_{\mathbb{X}}$, $x \approx y$ implies $f(x) \approx f(y)$.

We also define functions of class S^α for every multi-index α :

- f is of class $S^0(\Omega_{\mathbb{X}})$ if f is S -continuous on $\Omega_{\mathbb{X}}$;
- f is of class $S^\alpha(\Omega_{\mathbb{X}})$ if $\mathbb{D}^\alpha f \in S^0(\Omega_{\mathbb{X}})$.
- f is of class $S^\infty(\Omega_{\mathbb{X}})$ if $\mathbb{D}^\alpha f \in S^0(\Omega_{\mathbb{X}})$ for any standard multi-index α .

Notice that if $f \in S^\alpha(\Omega_{\mathbb{X}})$ for some standard multi-index α , then $f(x)$ is finite at all nearstandard $x \in \Omega_{\mathbb{X}}$.

In the study of S -continuous functions, we find it useful to introduce the following equivalence relation.

Definition 1.12. Let $f, g \in {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$. We say that $f \equiv_S g$ iff $(f - g)(x) \approx 0$ for all nearstandard $x \in \Omega_{\mathbb{X}}$. From the properties of \approx , it can be proved that \equiv_S is an equivalence relation. We will denote by π_S the projection from ${}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$ to the quotient space ${}^*\mathbb{R}^{\Omega_{\mathbb{X}}}/\equiv_S$, and will denote by $[f]_S$ the equivalence class of f with respect to \equiv_S .

The rest of this section is devoted to the proof that the quotient $S^\alpha(\Omega_{\mathbb{X}})/\equiv_S$ is real algebra isomorphic to the algebra of C^α functions over Ω . This result is a reformulation in the language of grid functions of some results by van den Berg [9] and by Wattenberg, Hanqiao, and St. Mary [32].

Lemma 1.13. For every standard multi-index α , $S^\alpha(\Omega_{\mathbb{X}})$ with pointwise sum and product is an algebra over ${}^*\mathbb{R}_{fin}$, and $S^\alpha(\Omega_{\mathbb{X}})/\equiv_S$ inherits a structure of real algebra from $S^\alpha(\Omega_{\mathbb{X}})$.

Proof. The only non-trivial assertion that needs to be verified is closure of $S^\alpha(\Omega_{\mathbb{X}})$ with respect to pointwise product. This property is a consequence of Proposition 2.6 of [9]. \square

Theorem 1.14. $S^0(\Omega_{\mathbb{X}})/\equiv_S$ is a real algebra isomorphic to $C^0(\Omega)$. The isomorphism is given by $i[f]_S = {}^\circ f$. The inverse of i is the function $i^{-1}(f) = [{}^*f]_S$.

Proof. If $f \in S^0(\Omega_{\mathbb{X}})$, then it is well-known that ${}^\circ f$ is a well-defined function and that ${}^\circ f \in C^0(\Omega)$. Surjectivity of ${}^\circ$ is a consequence of Lemma II.6 of [32]. Since

$$\ker({}^\circ) = \{f \in S^0(\Omega_{\mathbb{X}}) : f(x) \approx 0 \text{ for all finite } x \in \Omega_{\mathbb{X}}\} = [0]_S,$$

we deduce that i is injective and surjective. Since ${}^\circ(x + y) = {}^\circ x + {}^\circ y$ and ${}^\circ(xy) = {}^\circ x {}^\circ y$ for all $x, y \in {}^*\mathbb{R}_{fin}$, i_α is an isomorphism of real algebras. \square

We will now show that, for grid functions of class S^α , the finite difference operators \mathbb{D}_i^+ and \mathbb{D}_i^- assume the role of the usual partial derivative for C^α functions. In particular, these finite difference operators can be seen as generalized derivatives.

Theorem 1.15. *For all $1 \leq i \leq k$ and for all standard multi-indices α with $\alpha_i \geq 1$, the diagram*

$$\begin{array}{ccc} S^\alpha(\Omega_{\mathbb{X}}) & \xrightarrow{\mathbb{D}_i^+} & S^{\alpha-e_i}(\Omega_{\mathbb{X}}) \\ i \circ \pi_S \downarrow & & \downarrow i \circ \pi_S \\ C^\alpha(\Omega) & \xrightarrow{D_i} & C^{\alpha-e_i}(\Omega) \end{array}$$

and the diagram

$$\begin{array}{ccc} S^\alpha(\Omega_{\mathbb{X}}) & \xrightarrow{\mathbb{D}_i^-} & S^{\alpha-e_i}(\Omega_{\mathbb{X}}) \\ i \circ \pi_S \downarrow & & \downarrow i \circ \pi_S \\ C^\alpha(\Omega) & \xrightarrow{D_i} & C^{\alpha-e_i}(\Omega) \end{array}$$

commute.

Proof. By Theorem 1.14, if $f \in S^\alpha(\Omega_{\mathbb{X}}) \subseteq S^0(\Omega_{\mathbb{X}})$ then $(i_\alpha \circ \pi_S)(f) = {}^\circ f$ and, by Lemma II.7 of [32], ${}^\circ(\mathbb{D}_i^\pm f) = D_i {}^\circ f$. \square

By Theorem 1.15, the isomorphism i defined in Theorem 1.14 induces an isomorphism between $S^\alpha(\Omega_{\mathbb{X}})/\equiv_S$ and $C^\alpha(\Omega)$ as real algebras.

Corollary 1.16. *For any multi-index α , the isomorphism i restricted to $S^\alpha(\Omega_{\mathbb{X}})/\equiv_S$ induces an isomorphism between $S^\alpha(\Omega_{\mathbb{X}})/\equiv_S$ and $C^\alpha(\Omega)$ as real algebras.*

Thanks to this isomorphism, if $f \in S^\alpha(\Omega_{\mathbb{X}})$, we can identify the equivalence class $[f]_S$ with the standard function ${}^\circ f \in C^\alpha(\Omega)$.

2. GRID FUNCTIONS AS GENERALIZED DISTRIBUTIONS

In this section, we will study the relations between the space of grid functions and the space of distributions. In particular, we will prove that the space of grid functions can be seen as generalization of the space of distributions, and the operators \mathbb{D}^+ and \mathbb{D}^- coherently extend the distributional derivative to the space of grid functions.

In order to prove the above results, we start by defining a projection from an external ${}^*\mathbb{R}_{fin}$ -submodule of ${}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$ to the space of distributions. This projection is defined by duality with an external ${}^*\mathbb{R}_{fin}$ -algebra of grid functions that is a counterpart to the space of test functions.

Definition 2.1 (Algebra of test functions). *We define the algebra of test functions over $\Omega_{\mathbb{X}}$ as follows:*

$$\mathcal{D}_{\mathbb{X}}(\Omega_{\mathbb{X}}) = \{f \in S^\infty(\Omega_{\mathbb{X}}) : {}^\circ \text{supp } f \subset\subset \Omega\}.$$

The above definition provides a nonstandard counterpart of the usual space of smooth functions with compact support.

Lemma 2.2. *The isomorphism i defined in Theorem 1.14 induces an isomorphism between the real algebras $\mathcal{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})/\equiv_S$ and $\mathcal{D}(\Omega)$. The isomorphism preserves integrals, i.e. for all $\varphi \in \mathcal{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$, it holds the equality*

$$(2) \quad \circ \int_{\Omega_{\mathbb{X}}} \varphi d\mathbb{X}^k = \int_{\Omega} i[\varphi]_S dx.$$

Moreover, if $\varphi \in \mathcal{D}(\Omega)$, then ${}^*\varphi|_{\mathbb{X}} \in \mathcal{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$, so that $i^{-1}(\varphi) = [{}^*\varphi|_{\mathbb{X}}]_S \cap \mathcal{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$.

Proof. From Theorem 1.14, from Theorem 1.15 and from the definition of $\mathcal{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$, we can conclude that the hypothesis $\varphi \in \mathcal{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$ ensures that $i[\varphi] \in \mathcal{D}(\Omega)$. Since $\mathcal{D}_{\mathbb{X}}(\Omega_{\mathbb{X}}) \subset S^0(\Omega_{\mathbb{X}})$, injectivity of i is a consequence of Theorem 1.14.

Similarly, surjectivity of i can be deduced from Theorem 1.14 and from Theorem 1.15. In fact, suppose towards a contradiction that there exists $\psi \in \mathcal{D}(\Omega)$ such that $\psi \neq i[\varphi]$ for all $\varphi \in \mathcal{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$. Since $\psi \in C^0(\Omega)$, Theorem 1.14 ensures that there exists $\phi \in S^0(\Omega_{\mathbb{X}})$ with $i[\phi] = \psi$. If $\phi \notin S^\infty(\Omega_{\mathbb{X}})$, then for some standard multi-index α , $\mathbb{D}^\alpha \phi \notin S^0(\Omega_{\mathbb{X}})$, contradicting Theorem 1.15. As a consequence, i is an isomorphism between $\mathcal{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})/\equiv_S$ and $\mathcal{D}(\Omega)$.

Equality 2 is a consequence of the hypothesis $\circ \text{supp } \varphi \subset\subset \Omega$ and of Lemma 1.9.

Now let $\varphi \in \mathcal{D}(\Omega)$: by Theorem 1.14 and by Theorem 1.15, ${}^*\varphi|_{\mathbb{X}} \in S^\infty(\Omega_{\mathbb{X}})$. Let $A = \text{supp } \varphi$: since A is the closure of an open set, by Proposition 1.5 $\circ A_{\mathbb{X}} = A \subset\subset \Omega$, from which we deduce ${}^*\varphi|_{\mathbb{X}} \in \mathcal{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$. As a consequence, $i^{-1}(\varphi) = [{}^*\varphi|_{\mathbb{X}}]_S \cap \mathcal{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$, as we claimed. \square

The duality with respect to the space of test functions can be used to define an equivalence relation on the space of grid functions. This equivalence relation plays the role of a weak equality.

Definition 2.3. *Let $f, g \in {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$. We say that $f \equiv g$ iff for all $\varphi \in \mathcal{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$ it holds $\langle f, \varphi \rangle \approx \langle g, \varphi \rangle$. We will call π the projection from ${}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$ to the quotient ${}^*\mathbb{R}^{\Omega_{\mathbb{X}}}/\equiv$, and we will denote by $[f]$ the equivalence class of f with respect to \equiv .*

The new equivalence relation \equiv is coarser than \equiv_S .

Lemma 2.4. *For all $f, g \in {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$, $f \equiv_S g$ implies $f \equiv g$.*

Proof. We will show that $f \equiv_S g$ implies $\langle f - g, \varphi \rangle \approx 0$ for all $\varphi \in \mathcal{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$: by linearity of the hyperfinite sum, this result is equivalent to $f \equiv g$.

Let $\varphi \in \mathcal{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$, and let $\eta = \max_{x \in \text{supp } \varphi} \{|(f - g)(x)|\}$. The hypothesis that $f \equiv_S g$ and the hypothesis that $\circ \text{supp } \varphi$ is bounded are sufficient to ensure that $\eta \approx 0$. As a consequence, we have the following inequalities

$$|\langle f - g, \varphi \rangle| \leq \langle |f - g|, |\varphi| \rangle \leq |\eta| \int_{\Omega_{\mathbb{X}}} |\varphi(x)| d\mathbb{X}^k \approx 0,$$

that are sufficient to conclude the proof. \square

We can now define a duality pairing with respect to the inner product defined in 1.8.

Definition 2.5. For any $V \subseteq {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$, we define

$$V' = \{f \in {}^*\mathbb{R}^{\Omega_{\mathbb{X}}} : \langle g, f \rangle \text{ is finite for all } g \in V\}.$$

The ${}^*\mathbb{R}_{fin}$ -module

$$\mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}}) = \{f \in {}^*\mathbb{R}^{\Omega_{\mathbb{X}}} \mid \langle f, \varphi \rangle \text{ is finite for all } \varphi \in \mathcal{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})\}$$

is called the module of bounded grid functions.

The rest of this section is devoted to the proof that the quotient $\mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})/\equiv$ is real vector space isomorphic to the space of distributions $\mathcal{D}'(\Omega)$.

Lemma 2.6. For any $V \subseteq {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$, V' with pointwise sum and product is a module over ${}^*\mathbb{R}_{fin}$. Moreover, V'/\equiv inherits a structure of real vector space from V' .

Notice that, contrary to what happened for the space $S^0(\Omega_{\mathbb{X}})$, V' is not an algebra, since in general the hypothesis $f, g \in V'$ is not sufficient to ensure that $fg \in V'$.

The following characterization of bounded generalized distributions will be used in the proof of the isomorphism between the quotient of the module of the bounded generalized distributions and the space of distributions.

Lemma 2.7. The following are equivalent:

- (1) $f \in \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$;
- (2) $\langle f, \varphi \rangle \approx 0$ for all $\varphi \in \mathcal{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$ satisfying $\varphi(x) \approx 0$ for all $x \in \Omega_{\mathbb{X}}$.

Proof. (1) implies (2), by contrapositive. Suppose that $\langle f, \varphi \rangle \not\approx 0$ for some $\varphi \in \mathcal{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$ with $\varphi(x) \approx 0$ for all $x \in \Omega_{\mathbb{X}}$. Now take some $\psi \in \mathcal{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$ with $\psi(x) \geq n\varphi(x)$ for all $x \in \Omega_{\mathbb{X}}$ and for all $n \in \mathbb{N}$. From the inequality $\langle f, \psi \rangle \geq n\langle f, \varphi \rangle$ for all $n \in \mathbb{N}$, we deduce that $\langle f, \psi \rangle$ is infinite, i.e. that $f \notin \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$.

(2) implies (1), by contrapositive. Suppose that $\langle f, \varphi \rangle = M$ is infinite for some $\varphi \in \mathcal{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$. Since $\varphi/M \in \mathcal{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$ and $\varphi/M(x) \approx 0$ for all $x \in \Omega_{\mathbb{X}}$, we deduce that (2) does not hold. \square

From the above Lemma, we deduce that the action of a bounded generalized distribution over the space of test functions is continuous.

Corollary 2.8 (Continuity). If $\varphi, \psi \in \mathcal{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$ and $\varphi \equiv_S \psi$, then $\langle f, \varphi \rangle \approx \langle f, \psi \rangle$ for all $f \in \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$.

Proof. The hypotheses $\varphi, \psi \in \mathcal{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$ and $\varphi \equiv_S \psi$ imply $\varphi - \psi \in \mathcal{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$ and $(\varphi - \psi)(x) \approx 0$ for all $x \in \Omega_{\mathbb{X}}$. Then, by Lemma 2.7, we have $\langle f, \varphi - \psi \rangle \approx 0$ for all $f \in \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$, as we wanted. \square

We are now ready to prove that $\mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})/\equiv$ is isomorphic to the space of distributions over Ω .

Theorem 2.9. *The function $\Phi : (\mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})/\equiv) \rightarrow \mathcal{D}'(\Omega)$ defined by*

$$\langle \Phi([f]), \varphi \rangle_{\mathcal{D}(\Omega)} = {}^\circ \langle f, {}^*\varphi \rangle$$

is an isomorphism of real vector spaces.

Proof. At first, we will show that the definition of Φ does not depend upon the choice of the representative for $[f]$. Let $g, h \in [f]$: then, by definition of \equiv , ${}^\circ \langle g, \varphi \rangle = {}^\circ \langle h, \varphi \rangle$ for all $\varphi \in \mathcal{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$. By Lemma 2.2, for all $\varphi \in \mathcal{D}'(\Omega)$, ${}^*\varphi|_{\mathbb{X}} \in \mathcal{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$, so that if $g, h \in [f]$, then ${}^\circ \langle g, {}^*\varphi \rangle = {}^\circ \langle h, {}^*\varphi \rangle$ so that the definition of Φ is independent on the choice of the representative for $[f]$.

Lemma 2.8 ensures that for all $[f] \in \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})/\equiv$, $\Phi([f]) \in \mathcal{D}'_{\mathbb{X}}(\Omega)$, and in particular that $\Phi([f])$ is continuous.

We will prove by contradiction that Φ is injective. Suppose that $\langle \Phi([f]), \varphi \rangle = 0$ for all $\varphi \in \mathcal{D}(\Omega)$ and that $[f] \neq [0]$. The latter hypothesis implies that there exists $\psi \in \mathcal{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$ such that $\langle f, \psi \rangle \not\approx 0$. But, since ${}^*({}^\circ\psi)|_{\mathbb{X}} \equiv_S \psi$, by Corollary 2.8 we deduce

$$\langle \Phi([f]), {}^\circ\psi \rangle_{\mathcal{D}(\Omega)} = {}^\circ \langle f, {}^*({}^\circ\psi) \rangle = {}^\circ \langle f, \psi \rangle \neq 0,$$

contradicting the hypothesis $\langle \Phi([f]), \varphi \rangle = 0$ for all $\varphi \in \mathcal{D}(\Omega)$. As a consequence, Φ is injective.

Surjectivity of Φ is a consequence of Theorem 1 of [35]. \square

In view of the isomorphism Φ , from now on we will identify the equivalence class $[f]$ with the distribution $\Phi([f])$. Notice that if $f \in S^0(\Omega_{\mathbb{X}})$, this identification is coherent with $[f]_S$.

Corollary 2.10. *If $f \in S^0(\Omega_{\mathbb{X}})$, then $[f] = [f]_S = {}^\circ f$.*

Proof. Since f is S-continuous, by Lemma 1.9 and by Lemma 2.2 we have the equality

$$\int_{\Omega} {}^\circ f \varphi dx = {}^\circ \langle f, {}^*\varphi \rangle$$

for all $\varphi \in \mathcal{D}(\Omega)$, and this is sufficient to deduce the thesis. \square

Remark 2.11. *If $k \in \mathbb{N}$, define*

$$\mathcal{D}'_{\mathbb{X}}(\Omega, {}^*\mathbb{R}^k) = \left\{ f : \Omega_{\mathbb{X}} \rightarrow \mathbb{R}^k : f_i \in \mathcal{D}'_{\mathbb{X}}(\Omega) \text{ for all } 1 \leq i \leq k \right\}.$$

If $f \in \mathcal{D}'_{\mathbb{X}}(\Omega, {}^\mathbb{R}^k)$, then we can define a functional $[f]$ over the dual of the space of vector-valued test functions*

$$\mathcal{D}(\Omega, \mathbb{R}^k) = \left\{ \varphi : \Omega \rightarrow \mathbb{R}^k : \varphi_i \in \mathcal{D}(\Omega) \text{ for all } 1 \leq i \leq k \right\}$$

by posing $\langle [f], \varphi \rangle_{\mathcal{D}(\Omega, \mathbb{R}^k)} = \sum_{i=1}^k {}^\circ \langle f_i, {}^\varphi_i \rangle$ for all $\varphi \in \mathcal{D}(\Omega, \mathbb{R}^k)$. From Theorem 2.9, we deduce that the quotient of the ${}^*\mathbb{R}_{fin}$ -module $\mathcal{D}'_{\mathbb{X}}(\Omega, {}^*\mathbb{R}^k)$ with respect to \equiv is isomorphic to the real vector space of linear continuous functionals over $\mathcal{D}(\Omega, \mathbb{R}^k)$.*

Remark 2.12. *Theorem 2.9 can be used to define more general projections of nonstandard functions. For instance, if $f \in {}^*C^0({}^*\mathbb{R}, \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}}))$, then for all $T \in \mathbb{R}$ f induces a continuous linear functional $[f]$ over the space $C^0([0, T], \mathcal{D}'(\Omega))$ defined by the formula*

$$\int_0^T \langle [f], \varphi \rangle_{\mathcal{D}(\Omega)} dt = {}^\circ \left({}^* \int_0^T \langle f(t), {}^*\varphi(t) \rangle dt \right)$$

*for all $\varphi \in C^0([0, T], \mathcal{D}'(\Omega))$. Moreover, if $f \in {}^*C^1({}^*\mathbb{R}, \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}}))$, then $[f]$ allows for a weak derivative with respect to time: for all $T \in \mathbb{R}$, $[f]_t$ is the distribution that satisfies*

$$\int_0^T \langle [f]_t, \varphi \rangle_{\mathcal{D}(\Omega)} dt = -{}^\circ \left({}^* \int_0^T \langle f(t), {}^*\varphi(t) \rangle dt \right)$$

for all $\varphi \in C^1([0, T], \mathcal{D}'(\Omega))$.

2.1. Discrete derivative and distributional derivative. In this section, we will show that the finite difference operators \mathbb{D}_i^+ and \mathbb{D}_i^- generalize the distributional derivative to the setting of grid functions, i.e. that $[\mathbb{D}_i^\pm f] = D_i[f]$ for all $f \in \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$. For a matter of commodity, we will suppose that $\Omega_{\mathbb{X}} \subseteq \mathbb{X}$: the generalization to an arbitrary dimension can be deduced from the proof of Theorem 2.15 with an argument relying on Theorem 1.15.

Recall the discrete summation by parts formula: for all grid functions f and g and for all $a, b \in {}^*\mathbb{N}$ with $N^2 \leq a < b < N^2$ it holds the equality

$$\begin{aligned} \sum_{n=a}^b (f((n+1)\varepsilon) - f(n\varepsilon))g(n\varepsilon) &= f((b+1)\varepsilon)g((b+1)\varepsilon) - f(a\varepsilon)g(a\varepsilon) + \\ &\quad - \sum_{n=a}^b f((n+1)\varepsilon)(g((n+1)\varepsilon) - g(n\varepsilon)) \end{aligned}$$

that, in particular, implies

$$(3) \quad \langle \mathbb{D}f, \varphi \rangle = -\langle f(x + \varepsilon), \mathbb{D}\varphi \rangle$$

for all $f \in {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$ and for all $\varphi \in \mathcal{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$.

Inspired by the above formula, we will now prove that if we shift a bounded generalized distribution by an infinitesimal displacement, we still obtain the same generalized distribution.

Lemma 2.13. *Let $f \in {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$. Then $f(x) \in \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$ if and only if $f(x + \varepsilon) \in \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$. If $f(x) \in \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$ then, $[f(x)] = [f(x + \varepsilon)]$.*

Proof. The hypothesis that for all $\varphi \in \mathcal{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$ it holds ${}^\circ \text{supp } \varphi \subset \subset \Omega$ ensures the equality

$$\langle f(x), \varphi(x) \rangle = \langle f(x + \varepsilon), \varphi(x + \varepsilon) \rangle$$

from which we deduce the equivalence $f(x) \in \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$ if and only if $f(x + \varepsilon) \in \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$.

We will now prove that, $f(x) \in \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$, then $[f(x)] = [f(x + \varepsilon)]$. By equation 3, we have

$$(4) \quad \langle f(x + \varepsilon) - f(x), \varphi \rangle = -\langle f(x + \varepsilon), \varepsilon \mathbb{D}\varphi \rangle$$

for all $\varphi \in \mathcal{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$. Notice that $\varphi \in \mathcal{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$ implies that $\varepsilon \mathbb{D}\varphi \in \mathcal{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$ and $\varepsilon \mathbb{D}\varphi(x) \approx 0$ for all $x \in \Omega_{\mathbb{X}}$. Hence, by the hypothesis $f \in \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$ and by Lemma 2.7, we deduce that $\langle f(x + \varepsilon), \varepsilon \mathbb{D}\varphi \rangle \approx 0$ for all $\varphi \in \mathcal{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$. By equation 4, this is sufficient to deduce the equality $[f(x)] = [f(x + \varepsilon)]$. \square

As a consequence of the above Lemma, we can characterize a nonstandard counterpart of the shift operator.

Corollary 2.14. *Let $f \in \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$. For all n such that $n\varepsilon$ is finite, $[f(x \pm n\varepsilon)] = [f](x \pm \circ(n\varepsilon))$.*

We are now ready to prove that the finite difference operators generalize the distributional derivative.

Theorem 2.15. *The diagram*

$$\begin{array}{ccc} \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}}) & \xrightarrow{\mathbb{D}^+} & \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}}) \\ \Phi \circ \pi \downarrow & & \downarrow \Phi \circ \pi \\ \mathcal{D}'(\Omega) & \xrightarrow{D} & \mathcal{D}'(\Omega) \end{array}$$

and the diagram

$$\begin{array}{ccc} \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}}) & \xrightarrow{\mathbb{D}^-} & \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}}) \\ \Phi \circ \pi \downarrow & & \downarrow \Phi \circ \pi \\ \mathcal{D}'(\Omega) & \xrightarrow{D} & \mathcal{D}'(\Omega) \end{array}$$

commute.

Proof. We will prove that the first diagram commutes, as the proof for the second is similar.

Let $f \in \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$: we have the following equality chain

$$\langle D[f], \varphi \rangle_{\mathcal{D}(\Omega)} = -\langle [f], D\varphi \rangle_{\mathcal{D}(\Omega)} = -\circ \langle f, {}^*D^*\varphi \rangle.$$

By Theorem 1.15, ${}^*D^*\varphi \equiv_S \mathbb{D}^{\pm*}\varphi$ and, by Corollary 2.8,

$$\langle f, {}^*D^*\varphi \rangle \approx \langle f, \mathbb{D}^{\pm*}\varphi \rangle.$$

By the discrete summation by parts formula 3 and by Lemma 2.13 we have

$$\langle f, \mathbb{D}^{\pm*}\varphi \rangle \approx -\langle \mathbb{D}^{\pm}f, {}^*\varphi \rangle$$

from which we deduce

$$\langle D[f], \varphi \rangle_{\mathcal{D}(\Omega)} = \circ \langle [\mathbb{D}^{\pm}f], {}^*\varphi \rangle$$

for all $\varphi \in \mathcal{D}(\Omega)$. \square

By composing finite difference operators, we obtain the grid function counterpart of many differential operators, such as the grid gradient, the grid divergence and the grid Laplacian.

Definition 2.16 (Grid gradient, divergence and Laplacian). *If $f \in {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$, we define the forward and backward grid gradient of f as:*

$$\nabla_{\mathbb{X}}^{\pm} f = (\mathbb{D}_1^{\pm} f, \dots, \mathbb{D}_i^{\pm} f, \dots, \mathbb{D}_k^{\pm} f).$$

In a similar way, if $f : \Omega_{\mathbb{X}} \rightarrow {}^\mathbb{R}^k$, we define the forward and backward grid divergence as*

$$\operatorname{div}_{\mathbb{X}}^{\pm}(f(x, t)) = \sum_{i=1}^k \mathbb{D}_i^{\pm} f(x, t).$$

The grid Laplacian of $f \in {}^\mathbb{R}^{\Omega_{\mathbb{X}}}$ is defined as*

$$\Delta_{\mathbb{X}} f = \operatorname{div}_{\mathbb{X}}^{-}(\nabla_{\mathbb{X}}^{+}(f)) = \operatorname{div}_{\mathbb{X}}^{+}(\nabla_{\mathbb{X}}^{-}(f)) = \sum_{i=1}^k \mathbb{D}_i^{+} \mathbb{D}_i^{-} f.$$

In the sequel, we will mostly drop the symbol $+$ from the above definitions: for instance, we will write $\nabla_{\mathbb{X}}$ instead of $\nabla_{\mathbb{X}}^{+}$.

It is a consequence of Theorem 1.15 that, if $f \in S^{(1, \dots, 1)}(\Omega_{\mathbb{X}})$, then $\circ(\nabla_{\mathbb{X}}(f))$ is the usual gradient of $\circ f$, and similar results holds for $\nabla_{\mathbb{X}}^{-}$, $\operatorname{div}_{\mathbb{X}}$, $\operatorname{div}_{\mathbb{X}}^{-}$ and $\Delta_{\mathbb{X}}$. Moreover, by Theorem 2.15, the operators $\nabla_{\mathbb{X}}$ and $\nabla_{\mathbb{X}}^{-}$ satisfy the formula

$$\circ\langle \nabla_{\mathbb{X}} f, {}^*\varphi \rangle = \circ\langle \nabla_{\mathbb{X}}^{-} f, {}^*\varphi \rangle = -\langle [f], \operatorname{div} \varphi \rangle_{\mathcal{D}(\Omega)}$$

for all $f \in \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$ and for all functions $\varphi \in \mathcal{D}(\Omega, \mathbb{R}^k)$, and $\operatorname{div}_{\mathbb{X}}$ and $\operatorname{div}_{\mathbb{X}}^{-}$ satisfy the formula

$$\circ\langle \operatorname{div}_{\mathbb{X}} f, {}^*\varphi \rangle = \circ\langle \operatorname{div}_{\mathbb{X}}^{-} f, {}^*\varphi \rangle = -\langle [f], \nabla \varphi \rangle_{\mathcal{D}(\Omega)}$$

for all $f \in \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}}, {}^*\mathbb{R}^k)$ and for all $\varphi \in \mathcal{D}(\Omega)$. For the discrete Laplacian $\Delta_{\mathbb{X}}$, it holds

$$\circ\langle \Delta_{\mathbb{X}} f, {}^*\varphi \rangle = \langle [f], \Delta \varphi \rangle_{\mathcal{D}(\Omega)}$$

for all $f \in \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$ and for all $\varphi \in \mathcal{D}(\Omega)$.

2.2. Discrete chain rule for generalized distributions and the Schwartz impossibility theorem. For the usual distributions, it is well-known that some of the derivation rules that hold for smooth functions do not hold in general. In particular, it is a consequence of the impossibility theorem by Schwartz that no extension of the distributional derivative satisfies a product rule.

However, for the grid functions there are some discrete product rules that generalize the product rule for smooth functions. Indeed, the following identities can be established by a simple calculation.

Proposition 2.17 (Discrete product rules). *Let $f, g \in {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$. Then*

$$\begin{aligned} \mathbb{D}^{+}(f \cdot g)(x) &= \frac{f(x + \varepsilon)g(x + \varepsilon) - f(x)g(x)}{\varepsilon} \\ &= f(x + \varepsilon)\mathbb{D}^{+}g(x) + g(x)\mathbb{D}^{+}f(x) \\ &= f(x)\mathbb{D}^{+}g(x) + g(x + \varepsilon)\mathbb{D}^{+}f(x) \end{aligned}$$

and

$$\begin{aligned}\mathbb{D}^-(f \cdot g)(x) &= \frac{f(x)g(x) - f(x - \varepsilon)g(x - \varepsilon)}{\varepsilon} \\ &= f(x)\mathbb{D}^-g(x) + g(x - \varepsilon)\mathbb{D}^-f(x) \\ &= f(x - \varepsilon)\mathbb{D}^-g(x) + g(x)\mathbb{D}^-f(x).\end{aligned}$$

Example 2.18 (Derivative of the sign function and the product rule). *For an in-depth discussion of this example and of the limitations in the definition of a product rule for the distributional derivative, we refer to [54]. Consider the following representative of the sign function*

$$u(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0. \end{cases}$$

For this function u , $u^2 = 1$ and $u^3 = u$, but the distributional derivative $u_x = 2\delta_0$ is different from $(u^3)_x = 3u^2u_x = 3u_x = 6\delta_0$. So, even if u^2 is smooth, the product rule does not hold.

If we regard u as a grid function, however, it is easy to see that $u \in \mathcal{D}'_{\mathbb{X}}(\mathbb{X})$, and with a simple calculation we obtain:

$$(5) \quad \mathbb{D}u(x) = \begin{cases} 2\varepsilon^{-1} & \text{if } x = -\varepsilon \\ 0 & \text{otherwise.} \end{cases}$$

Notice also that $[\mathbb{D}u] = 2\delta_0$, as we expected from Theorem 2.15. Applying one of the chain rule formulas of Lemma 2.17 and taking into account that $u^2(x) = 1$ for all $x \in \mathbb{X}$, we obtain

$$\begin{aligned}\mathbb{D}u^3(x) &= u(x)\mathbb{D}u^2(x) + u^2(x + \varepsilon)\mathbb{D}u(x) \\ &= u(x)(u(x)\mathbb{D}(x) + u(x + \varepsilon)\mathbb{D}u(x)) + \mathbb{D}u(x) \\ &= \mathbb{D}u(x)(2 + u(x)u(x + \varepsilon))\end{aligned}$$

so that

$$\mathbb{D}u^3(x) = \begin{cases} \mathbb{D}u(-\varepsilon) = 2\varepsilon^{-1} & \text{if } x = -\varepsilon \\ 0 & \text{otherwise,} \end{cases}$$

in agreement with 5.

We can summarize the results obtained so far as follows: the space of grid functions

- is a vector space over ${}^*\mathbb{R}$ that extends the space of distributions in the sense of Theorem 2.9;
- has a well-defined pointwise multiplication that extends the one defined for S^0 functions;
- has a derivative \mathbb{D} that generalizes the distributional derivative and for which the discrete version of the chain rule established in Proposition 2.17 holds.

These properties are the nonstandard, discrete counterparts to the ones itemized in the impossibility theorem by Schwartz [49]. As a consequence,

the space of grid functions can be seen as a non-trivial generalization of the space of distributions, as we claimed at the beginning of this section.

We will complete our discussion about the relation of the space of grid functions and the space of distributions by showing that the space of distributions can be embedded, albeit in a non-canonical way, in the space of grid functions. Notice that we cannot ask to this embedding to be fully coherent with derivatives: in fact, there is already an infinitesimal discrepancy between the usual derivative and the discrete derivative in the set of polynomials: the derivative of x^2 is $2x$, but $\mathbb{D}x^2 = 2x + \varepsilon$. However, as shown in Theorem 1.15, for all $f \in C^n$, $D^n f = [\mathbb{D}^n(*f|_{\mathbb{X}})]$. In fact, the canonical linear embedding $l : C^0(\mathbb{R}) \hookrightarrow S^0(\mathbb{X})$ given by $l(f) = *f|_{\mathbb{X}}$ does not preserve derivatives, but it has the weaker property

$$(6) \quad l(f') \equiv \mathbb{D}(l(f)).$$

This will be the weaker coherence request that we will impose on the embedding from the space of distributions to the space of grid functions.

Theorem 2.19. *Let $\{\psi_n\}_{n \in \mathbb{N}}$ be a partition of unity, and let H be a Hamel basis for $\mathcal{D}'(\mathbb{R})$. There is a linear embedding $l : \mathcal{D}'(\mathbb{R}) \rightarrow \mathcal{D}'_{\mathbb{X}}(\mathbb{X})$, that depends on $\{\psi_n\}_{n \in \mathbb{N}}$ and H , that satisfies the following properties:*

- (1) $\Phi \circ l = id$;
- (2) the product over $\mathcal{D}'_{\mathbb{X}}(\mathbb{X}) \times \mathcal{D}'_{\mathbb{X}}(\mathbb{X})$ generalizes the pointwise product over $C^0(\mathbb{R}) \times C^0(\mathbb{R})$;
- (3) the derivative \mathbb{D} over $\mathcal{D}'_{\mathbb{X}}(\mathbb{X})$ extends the distributional derivative in the sense of equation 6;
- (4) the chain rule for products holds in the form established in Lemma 2.17.

Proof. We will define l over H and extend it to all of $\mathcal{D}'(\mathbb{R})$ by linearity. Let $T \in H$. From the representation theorem of distributions (see for instance [52]), we obtain

$$(7) \quad T = \sum_{n \in \mathbb{N}} T\psi_n = \sum_{n \in \mathbb{N}} D^{a_n} f_n$$

with $f_n \in C^0(\mathbb{R})$ and $\text{supp}(D^{a_n} f_n) \subseteq \text{supp} \psi_n$ for all $n \in \mathbb{N}$. Moreover, the sum is locally finite and for all $\varphi \in \mathcal{D}(\Omega)$ there exists a finite set $I_\varphi \subset \mathbb{N}$ such that

$$(8) \quad \langle T, \varphi \rangle_{\mathcal{D}(\Omega)} = \langle \sum_{i \in I_\varphi} D^{a_i} f_i, \varphi \rangle_{\mathcal{D}(\Omega)}.$$

Let $\{\phi_n\}_{n \in {}^*\mathbb{N}}$ be the nonstandard extension of the sequence $\{\psi_n\}_{n \in \mathbb{N}}$, and let $\{b_n\}_{n \in {}^*\mathbb{N}}$ be the nonstandard extension of the sequence $\{a_n\}_{n \in \mathbb{N}}$. By transfer, from the representation 7 we obtain

$$(9) \quad {}^*T = \sum_{n \in {}^*\mathbb{N}} {}^*T\phi_n = \sum_{n \in {}^*\mathbb{N}} {}^*D^{b_n} g_n$$

with $g_i \in {}^*C^0(\mathbb{R})$ and $\text{supp}(D^{b_n}g_n) \subseteq \text{supp}\psi_n$ for all $n \in {}^*\mathbb{N}$. We may also assume that the representation 9 has the following properties:

- (1) $b_n = \min \{m \in {}^*\mathbb{N} : {}^*T\phi_n = {}^*D^m f \text{ with } f \in {}^*C^0(\mathbb{R})\}$ for all $n \in {}^*\mathbb{N}$
- (2) if ${}^*T\phi_n = {}^*D^{b_n}g = {}^*D^{b_n}h$ with $g, h \in {}^*C^0(\mathbb{R})$, then $g - h$ is a polynomial of a degree not greater than $b_n - 1$;
- (3) if n is finite and ${}^*T\phi_n = {}^*D^{b_n}g_n$, then $g_n = {}^*f_n$ and $b_n = a_n$, where f_n and a_n satisfy $T\psi_n = D^{a_n}f_n$.

For $T \in H$, we define

$$l(T) = \sum_{n \in {}^*\mathbb{N}: b_n \leq N} \mathbb{D}^{b_n}(g_n|_{\mathbb{X}}),$$

and we extend l to $\mathcal{D}'(\mathbb{R})$ by linearity. Notice that l does not depend on the choice of the functions $\{g_n\}_{n \in {}^*\mathbb{N}}$. In fact, suppose that ${}^*T\phi_n = {}^*D^{b_n}g = {}^*D^{b_n}h$ with $g, h \in {}^*C^0(\mathbb{R})$. By property (2) of the representation 9, $g - h$ is a polynomial of a degree not greater than $b_n - 1$. Recall that, if $p \in {}^*\mathbb{R}^{\mathbb{X}}$ is a polynomial of degree at most $b_n - 1$, then $\mathbb{D}^{b_n}p = 0$. As a consequence, $\mathbb{D}^{b_n}(g|_{\mathbb{X}}) = \mathbb{D}^{b_n}(h|_{\mathbb{X}})$, as we wanted.

We will now show that, for all $T \in H$, $\langle \Phi([l(T)]), \varphi \rangle_{\mathcal{D}(\Omega)} = \langle T, \varphi \rangle_{\mathcal{D}(\Omega)}$ for all $\varphi \in \mathcal{D}'(\mathbb{R})$. This and linearity of l entail that $\Phi \circ l = \text{id}$. Let $\varphi \in \mathcal{D}(\mathbb{R})$, and let $I_\varphi \subset \mathbb{N}$ a finite set such that equality 8 holds. We claim that whenever $i \notin I_\varphi$, then $\langle \mathbb{D}^{b_i}(g_i|_{\mathbb{X}}), {}^*\varphi \rangle = 0$. In fact, if $i \notin I_\varphi$ is finite, then by formula 8 and by property (3) of the representation 9 we have

$${}^\circ \langle \mathbb{D}^{b_i}(g_i|_{\mathbb{X}}), \varphi \rangle = {}^\circ \langle \mathbb{D}^{a_i}({}^*f_i|_{\mathbb{X}}), \varphi \rangle = \langle D^{a_i}f_i, \varphi \rangle_{\mathcal{D}(\Omega)} = 0.$$

We want to show that $\langle \mathbb{D}^{b_i}(g_i|_{\mathbb{X}}), {}^*\varphi \rangle = 0$ also when i is infinite. Notice that if $x \in {}^*\mathbb{R}_{fin}$, then for sufficiently large $n \in \mathbb{N}$ it holds $x \notin \text{supp}\phi_n$: otherwise, we would also have ${}^\circ x \in \text{supp}\psi_n$ for arbitrarily large n , against the fact that for all $x \in {}^*\mathbb{R}_{fin}$, ${}^\circ x \in \text{supp}\phi_n$ only for finitely many n . As a consequence, $\text{supp}\phi_i \cap {}^*\mathbb{R}_{fin} = \emptyset$, and by the inclusion $\text{supp}(D^{b_i}g_i) \subseteq \text{supp}\phi_i$, then also $\text{supp}(D^{b_i}g_i) \cap {}^*\mathbb{R}_{fin} = \emptyset$. Taking into account property (2) of the representation 9, we deduce that the restriction of g_i to ${}^*\mathbb{R} \setminus \text{supp}(D^{b_i}g_i)$ is a polynomial p of degree at most $b_n - 1$. We have already observed that $\mathbb{D}^{b_i}p = 0$ and, as a consequence, ${}^\circ \langle \mathbb{D}^{b_i}(g_i|_{\mathbb{X}}), \varphi \rangle = 0$.

We then have the following equality:

$$\langle l(T), {}^*\varphi \rangle = \langle \sum_{i \in I_\varphi} \mathbb{D}^{a_i}({}^*f_i|_{\mathbb{X}}), {}^*\varphi \rangle.$$

By Theorem 2.15, we obtain

$$\langle l(T), {}^*\varphi \rangle = \langle \sum_{i \in I_\varphi} \mathbb{D}^{a_i}({}^*f_i|_{\mathbb{X}}), {}^*\varphi \rangle = \langle \sum_{i \in I_\varphi} D^{a_i}f_i, \varphi \rangle_{\mathcal{D}(\Omega)} = \langle T, \varphi \rangle_{\mathcal{D}(\Omega)},$$

that is sufficient to conclude that $\Phi([l(T)]) = T$.

Assertion (2) is a consequence of Lemma 1.13, assertion (3) is a consequence of Theorem 2.15, and assertion (4) is a consequence of Proposition 2.17. \square

3. GRID FUNCTIONS AS ${}^*L^p$ FUNCTIONS AND AS PARAMETRIZED MEASURES

The main goal of this section is to show that there is an external ${}^*\mathbb{R}_{fin}$ -submodule of the space of grid functions whose elements correspond to Young measures, and that this correspondence is coherent with the projection Φ defined in Theorem 2.9. Moreover, we will show how this correspondence can be generalized to arbitrary grid functions. Before we prove these results, we find it useful to discuss some properties of grid functions when they are interpreted as ${}^*L^p$ functions. These properties will be used also in Section 4, when we will discuss the grid function formulation of partial differential equations.

Recall that for all $1 \leq p \leq \infty$, a function $f \in L^p(\Omega)$ induces a distribution $T_f \in \mathcal{D}'(\Omega)$ defined by

$$\langle T_f, \varphi \rangle_{\mathcal{D}'(\Omega)} = \int_{\Omega} f \varphi dx$$

for all $\varphi \in \mathcal{D}(\Omega)$. As a consequence, by identifying f with T_f we have the inclusions $L^p(\Omega) \subset \mathcal{D}'(\Omega)$ for all $1 \leq p \leq \infty$. Since Φ is surjective, we expect that for all $f \in L^p(\Omega)$ there exists $g \in \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$ satisfying $[g] = T_f$. In this case, we will often write $[g] = f$ and $[g] \in L^p(\Omega)$. If $[g] \in L^p(\Omega)$, thanks to the Riesz representation theorem, we can think of $[g]$ either as a functional acting on $L^{p'}(\Omega)$, or as a member of an equivalence class of $L^p(\Omega)$ functions. To our purposes, we find it more convenient to treat $[g]$ as a function. With this interpretation, if $f = [g]$ and $f \in L^p(\Omega)$, then it holds the equality $f(x) = [g](x)$ for almost every $x \in \Omega$.

3.1. Grid functions as ${}^*L^p$ functions. If $f \in {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$, then we can identify f with a piecewise constant function defined on all of ${}^*\mathbb{R}^k$. Among many different extensions, we choose to represent f by the function \hat{f} defined by

$$\hat{f}(x) = \begin{cases} f((n_1, n_2, \dots, n_k)\varepsilon) & \text{if } n_i\varepsilon \leq x_i < (n_i + 1)\varepsilon \text{ for all } 1 \leq i \leq k \\ 0 & \text{if } |x_i| > N \text{ for some } 1 \leq i \leq k, \end{cases}$$

with the agreement that $f((n_1, n_2, \dots, n_k)\varepsilon) = 0$ if $(n_1, n_2, \dots, n_k)\varepsilon \notin \Omega_{\mathbb{X}}$.

If f is a grid function, the function \hat{f} is an internal * simple function and, as such, it belongs to ${}^*L^p(\mathbb{R}^k)$ for all $1 \leq p \leq \infty$. The integral of \hat{f} is related with the grid integral of f by the following formula:

$${}^*\int_{{}^*\mathbb{R}^k} \hat{f} dx = \int_{\Omega_{\mathbb{X}}} f(x) d\mathbb{X}^k = \varepsilon^k \sum_{x \in \Omega_{\mathbb{X}}} f(x).$$

As a consequence, the ${}^*L^p$ norm of \widehat{f} can be expressed by

$$\|\widehat{f}\|_p^p = \varepsilon^k \sum_{x \in \Omega_{\mathbb{X}}} |f(x)|^p \text{ if } 1 \leq p < \infty, \text{ and } \|\widehat{f}\|_{\infty} = \max_{x \in \Omega_{\mathbb{X}}} |f(x)|.$$

Moreover, notice that if $f \in {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$, then ${}^\circ\text{supp } \widehat{f} \subseteq {}^\circ\text{supp } \widehat{\chi_{\Omega_{\mathbb{X}}}} = \overline{\Omega}$. If we define $\widehat{\Omega} = \text{supp } \widehat{\chi_{\Omega_{\mathbb{X}}}}$, then from the above inclusion we can write $\widehat{f} \in {}^*L^p(\widehat{\Omega})$ for all $1 \leq p \leq \infty$. By identifying $f \in {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$ with \widehat{f} , for all $1 \leq p \leq \infty$ the space of grid functions is identified with a subspace of ${}^*L^p(\widehat{\Omega})$ which is closed with respect to the ${}^*L^p$ norm. Since $\widehat{\Omega}$ is * bounded in ${}^*\mathbb{R}^k$, for $1 \leq p \leq \infty$ we have the usual relations between the ${}^*L^p$ norms of $f \in {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$:

$$\|\widehat{f}\|_1 \leq \|\widehat{f}\|_p \leq \|\widehat{f}\|_{\infty}.$$

From now on, when there is no risk of confusion, we will often abuse the notation and write f instead of \widehat{f} .

We begin our study of grid functions as ${}^*L^p$ functions by showing that if a grid function f has finite ${}^*L^p$ norm for some $1 \leq p \leq \infty$, then $f \in \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$ and, as a consequence, $[f]$ is a well-defined distribution.

Lemma 3.1. *If $\|f\|_p \in {}^*\mathbb{R}_{fin}$ for some $1 \leq p \leq \infty$, then $f \in \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$.*

Proof. Notice that $\mathcal{D}_{\mathbb{X}}(\Omega_{\mathbb{X}}) \subset {}^*L^p(\widehat{\Omega})$ for all $1 \leq p \leq \infty$ and, for any $\varphi \in \mathcal{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$, $\|\varphi\|_p \in {}^*\mathbb{R}_{fin}$ for all $1 \leq p \leq \infty$. By the discrete Hölder's inequality

$$|\langle f, \varphi \rangle| \leq \|f\varphi\|_1 \leq \|f\|_p \|\varphi\|_{p'}$$

so that if $\|f\|_p \in {}^*\mathbb{R}_{fin}$, then $\langle f, \varphi \rangle \in {}^*\mathbb{R}_{fin}$ for all $\varphi \in \mathcal{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$, as desired. \square

From the previous Lemma we deduce that, if the L^p norm of the difference of two grid functions f and g is infinitesimal, then $f \equiv g$.

Corollary 3.2. *Let $f, g \in {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$. If $\|f - g\|_p \approx 0$ for some $1 \leq p \leq \infty$, then $f \equiv g$.*

Proof. If $\|f - g\|_p \approx 0$, then by Lemma 3.1

$$\langle f - g, \varphi \rangle \leq \|f - g\|_p \|\varphi\|_{p'} \approx 0$$

for all $\varphi \in \mathcal{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$. As a consequence, $f \equiv g$. \square

Notice that the other implication does not hold, in general. As an example, consider the grid function $f(n\varepsilon) = (-1)^n$. Since $\langle f, \varphi \rangle \approx 0$ for all $\varphi \in \mathcal{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$, we deduce that $[f] = 0$, but $\|f\|_p = 1$ for all $1 \leq p \leq \infty$. Notice also that $\|f\|_p$ is finite, but $\|\widehat{f} - {}^*g\|_p \not\approx 0$ for all $g \in L^p(\Omega)$ and for all $1 \leq p \leq \infty$.

In the next section, we will show that the hypothesis $\|f\|_{\infty} \in {}^*\mathbb{R}_{fin}$ is sufficient to ensure that $[f] \in L^{\infty}(\Omega)$. If $1 \leq p < \infty$, however, the hypothesis $\|f\|_p \in {}^*\mathbb{R}_{fin}$ is not sufficient to imply that $[f] \in L^p(\Omega)$. An example is given

by $N\chi_0 \in {}^*\mathbb{R}^{\mathbb{X}}$, a representative of the Dirac distribution centred at 0. It can be calculated that

$$\|N\chi_0\|_1 = \varepsilon N = 1,$$

but $[N\chi_0] = \delta_0 \notin L^p(\mathbb{R})$ for any p . In general, whenever $[f] \in L^p(\Omega)$, it holds the inequality $\|f\|_p \geq \|[f]\|_p$.

Proposition 3.3. *For all $f \in {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$ and for all $1 \leq p \leq \infty$, if $[f] \in L^p(\Omega)$, then*

- (1) *if $\|[f]\| \in L^p(\Omega)$, then $\|[f]\| \geq |[f]|$ a.e. in Ω ;*
- (2) *$\circ\|f\|_p \geq \|[f]\|_p$.*

Proof. Define $f^+(x) = \max\{f(x), 0\}$ and $f^-(x) = \min\{f(x), 0\}$, so that $f = f^+ + f^-$ and $|f|^p = |f^+|^p + |f^-|^p$ for all $1 \leq p < \infty$. If $\|[f]\| \in L^p(\Omega)$, then $[f^+]$ and $[f^-]$ $\in L^p(\Omega)$ and, by linearity of Φ ,

$$\|[f]\|(x) = [f^+](x) - [f^-](x) \geq [f^+](x) + [f^-](x) = [f](x)$$

for a.e. $x \in \Omega$.

Let $f \in {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$ and suppose that $[f] \in L^p(\Omega)$ with $p < \infty$. If either $|f^+| \notin \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$, $|f^+|^p \notin \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$, $|f^-| \notin \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$ or $|f^-|^p \notin \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$ then by Lemma 3.1 we would have $|f|^p \notin \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$ and, as a consequence,

$$\|f\|_p^p = \||f|^p\|_1 \notin {}^*\mathbb{R}_{fin},$$

so that inequality (2) would hold. Suppose then that $|f^+| \in \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$, $|f^+|^p \in \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$, $|f^-| \in \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$ and $|f^-|^p \in \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$. As a consequence, both $|f| \in \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$ and $|f|^p \in \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$. If $\|[f]\| \in L^p(\Omega)$, then (2) is a consequence of (1). The only case left is $\|[f]\| \notin L^p(\Omega)$.

For a matter of commodity, let $g = [f]$, and let $g^+(x) = \max\{g(x), 0\}$ and $g^-(x) = \min\{g(x), 0\}$. Since

$$[f^+] + [f^-] = [f] = g^+ + g^- \text{ in } \mathcal{D}'(\Omega),$$

we deduce that

$$[f^+] - g^+ = -([f^-] - g^-).$$

Since $[f^+] \notin L^p(\Omega)$, then also $[f^+] - g^+ \notin L^p(\Omega)$. Let $K = \text{supp}([f^+] - g^+)$: then for all $\varphi \in \mathcal{D}(\Omega)$ with $\text{supp } \varphi \subset K$ and with $\varphi(x) \geq 0$ for all $x \in \Omega$,

$$0 \leq \langle [f^+] - g^+, \varphi \rangle_{\mathcal{D}(\Omega)} = \circ\langle f^+, * \varphi \rangle - \int_{\Omega} g^+ \varphi dx.$$

Similarly,

$$0 \leq -\langle [f^-] - g^-, \varphi \rangle_{\mathcal{D}(\Omega)} = \circ\langle |f^-|, * \varphi \rangle - \int_{\Omega} |g^-| \varphi dx.$$

From the arbitrariness of φ , we deduce $\|f\chi_{K_{\mathbb{X}}}\|_p \geq \|g\chi_K\|_p$. Since $K = \text{supp}([f^+] - g^+)$, we also have

$$\|([f^+] - g^+)\chi_{\Omega \setminus K}\|_p = \|([f^-] - g^-)\chi_{\Omega \setminus K}\|_p = \|0\|_p = 0,$$

from which we conclude that (2) indeed holds.

Suppose now that $[f] \in L^\infty(\Omega)$. If $\|f\|_\infty \notin {}^*\mathbb{R}_{fin}$, then inequality (2) holds. If $\|f\|_\infty \in {}^*\mathbb{R}_{fin}$, let $c_f \in {}^*\mathbb{R}^{\Omega_\mathbb{X}}$ satisfy $c_f(x) = \|f\|_\infty$ for all $x \in \Omega_\mathbb{X}$. Then $[c_f](x) = {}^\circ\|f\|_\infty$ for all $x \in \Omega$, so that $[c_f] \in L^\infty(\Omega)$. Since $c_f(x) \geq \max\{f^+(x), |f^-(x)|\}$ for all $x \in \Omega_\mathbb{X}$, then also $[c_f](x) \geq [f](x)$ for all $x \in \Omega_\mathbb{X}$. This is sufficient to conclude that inequality (2) holds. \square

If $[f] \in L^p(\Omega_\mathbb{X})$ and ${}^\circ\|f\|_p > \|[f]\|_p$, then f features some oscillations that are compensated by the linearity of Φ . In this case, we can interpret f as the representative of a weak or (weak- \star when $p = \infty$) limit of a sequence of functions whose L^p norm is uniformly bounded by ${}^\circ\|f\|_p$. In the next section, we will see how the behaviour of this weak- \star limit can be described by a parametrized measure associated to f .

If $\|f\|_p \notin {}^*\mathbb{R}_{fin}$ but nevertheless $[f] \in L^p(\Omega)$, then f also features concentrations that are compensated by the linearity of Φ . An example is given by the function $f = \mathbb{D}\chi_0 = N\chi_{-\varepsilon} - N\chi_0$. The ${}^*L^p$ norm of f is $\|f\|_p = 2N^{p-1/p}$ for $p \neq \infty$ and N for $p = \infty$; however, from Theorem 2.15, we deduce that $[f] = D[\chi_0] = 0$. In the next section, we will discuss how these concentrations affect the parametrized measure associated to f .

We will now address the coherence between the nonstandard extension of a L^2 function and its projection in the space of grid functions. These technical results will be used in Section 4.

Definition 3.4. Let $P : {}^*L^2(\widehat{\Omega}) \rightarrow {}^*\mathbb{R}^{\Omega_\mathbb{X}}$ be the ${}^*L^2$ projection over the closed subspace ${}^*\mathbb{R}^{\Omega_\mathbb{X}}$. Recall that $P(f)$ is the unique grid function satisfying

$$\langle P(f), g \rangle = {}^*\int_{\Omega_\mathbb{X}} f(x)\widehat{g}(x)dx$$

for all $g \in {}^*\mathbb{R}^{\Omega_\mathbb{X}}$.

Lemma 3.5. For all $f \in C^0(\Omega)$, $P({}^*f) \in S^0(\Omega_\mathbb{X})$ and ${}^*f(x) \approx P({}^*f)(x)$ for all $x \in \Omega_\mathbb{X}$.

Proof. Let $f \in C^0(\Omega)$. Since for all $g \in {}^*\mathbb{R}^{\Omega_\mathbb{X}}$ we have the equality

$$\langle P({}^*f), g \rangle = {}^*\int_{\Omega_\mathbb{X}} {}^*f(x)\widehat{g}(x)dx,$$

by choosing $g = \varepsilon^{-k}\chi_y$, we obtain

$$P({}^*f)(y) = \langle P({}^*f), \widehat{\varepsilon^{-k}\chi_y} \rangle = \varepsilon^{-k} {}^*\int_{[y, y+\varepsilon]^k} {}^*f(x)dx$$

for all $y \in \Omega_\mathbb{X}$. Since

$$\min_{x \in [y, y+\varepsilon]^k} \{{}^*f(x)\} \leq \varepsilon^{-k} {}^*\int_{[y, y+\varepsilon]^k} {}^*f(x)dx \leq \max_{x \in [y, y+\varepsilon]^k} \{{}^*f(x)\},$$

by S-continuity of *f , we deduce the thesis. \square

Lemma 3.6. For all $f \in L^2(\Omega)$, $[P({}^*f)] = f$.

Proof. For all $\varphi \in \mathcal{D}'(\Omega)$ we have

$$\langle P(*f), {}^*\varphi|_{\mathbb{X}} \rangle = {}^*\int_{{}^*\Omega} {}^*f \widehat{{}^*\varphi|_{\mathbb{X}}} dx$$

and, by S-continuity of ${}^*\varphi$,

$${}^*\int_{{}^*\Omega} {}^*f \widehat{{}^*\varphi|_{\mathbb{X}}} dx \approx {}^*\int_{{}^*\Omega} {}^*f {}^*\varphi dx = \int_{\Omega} f \varphi dx.$$

This implies $[P(*f)] = f$. \square

The above Lemma can be sharpened under the hypothesis that Ω has finite measure.

Lemma 3.7. *Let $\mu_L(\Omega) < +\infty$. For all $f \in L^2(\Omega)$, $\|*f - P(*f)\|_2 \approx 0$.*

Proof. Let $f \in L^2(\Omega)$, and let $r = *f - P(*f)$. By the properties of the $*L^2$ projection, we have

$$(10) \quad \|*f\|_2 = \|P(*f)\|_2 + \|r\|_2.$$

By the nonstandard Luzin's Theorem, there exists a $*\text{compact set } K \subseteq {}^*\Omega$ that satisfies ${}^*\mu_L({}^*\Omega \setminus K) \approx 0$ and $\|r\chi_K\|_2 \approx 0$. Since ${}^*\mu_L({}^*\Omega \setminus K) \approx 0$ and since $f \in L^2(\Omega)$, we have also $\|*f\chi_K\|_2 \approx \|*f\|_2$ and, as a consequence,

$$\|*f\|_2 \approx \|*f\chi_K\|_2 = \|P(*f)\chi_K\|_2 + \|r\chi_K\|_2 \approx \|P(*f)\chi_K\|_2.$$

From the inequality chain

$$\|*f\|_2 \approx \|P(*f)\chi_K\|_2 \leq \|P(*f)\|_2 \leq \|*f\|_2$$

we deduce that $\|*f\|_2 \approx \|P(*f)\|_2$ that, by equality 10, implies $\|*f - P(*f)\|_2 \approx 0$, as we wanted. \square

The previous Lemma suggests a definition of nearstandardness that will be useful in the sequel of the paper.

Definition 3.8. *Let $\mu_L(\Omega) < +\infty$. We will say that $f \in {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$ is nearstandard in $L^2(\Omega)$ iff there exists $g \in L^2(\Omega)$ such that $\|f - P(*g)\|_2 \approx 0$.*

Notice that, thanks to Corollary 3.2 and to Lemma 3.7, f is nearstandard in $L^2(\Omega)$ if and only if $[f] \in L^2(\Omega)$ and $\|f - P(*[f])\|_2 \approx 0$.

We conclude the study of the properties of grid functions as $*L^p$ functions by discussing the generalization of an embedding due to Robinson and Bernstein

$$L^2(\Omega) \subset V \subset {}^*L^2(\Omega),$$

where V is a vector space of a hyperfinite dimension (for the details, we refer to [11, 25]). In our case, by considering the embedding l of the space of distributions to the space of grid functions defined in Theorem 2.19 and by modifying the extension of f to \widehat{f} , we will obtain the inclusions

$$L^p(\Omega) \subset \mathcal{D}'(\Omega) \subset {}^*\mathbb{R}^{\Omega_{\mathbb{X}}} \subset {}^*L^p(\Omega)$$

for all $1 \leq p \leq \infty$.

Proposition 3.9. *Let l be defined as in the proof of Theorem 2.19. There is an embedding $l' : {}^*\mathbb{R}^{\Omega_{\mathbb{X}}} \rightarrow \bigcap_{1 \leq p \leq \infty} {}^*L^p(\Omega)$ such that*

$$(11) \quad {}^*\int_{{}^*\mathbb{R}^k} (l' \circ l)(f) {}^*\varphi dx \approx \int_{\mathbb{R}^k} f \varphi dx$$

for all $1 \leq p \leq \infty$, for all $f \in L^p(\Omega)$ and for all $\varphi \in \mathcal{D}(\Omega)$. As a consequence, if we identify $\mathcal{D}'(\Omega)$ with $l(\mathcal{D}'(\Omega)) \subseteq {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$ and ${}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$ with $l'({}^*\mathbb{R}^{\Omega_{\mathbb{X}}}) \subseteq {}^*L^p(\Omega)$, we have the inclusions

$$L^p(\Omega) \subset \mathcal{D}'(\Omega) \subset {}^*\mathbb{R}^{\Omega_{\mathbb{X}}} \subset {}^*L^p(\Omega)$$

for all $1 \leq p \leq \infty$.

Proof. Define l' by $l'(f) = \widehat{f}\chi_{*\Omega}$ for all $f \in {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$. Since $l'(f)$ is an internal * simple function, it belongs to ${}^*L^p(\Omega)$ for all $1 \leq p \leq \infty$. We will now prove that, for this choice of l' , equality 11 holds.

Notice that for all $f \in {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$, if $l'(f)(x) \neq \widehat{f}(x)$, then $x \in {}^*\Omega \setminus \widehat{\Omega}$ or $x \in \widehat{\Omega} \setminus {}^*\Omega$. By the definition of $\widehat{\Omega}$, this entails ${}^\circ x \in \partial\Omega$. In particular, if $\varphi \in \mathcal{D}(\Omega)$, then ${}^\circ x \notin \text{supp } \varphi$. As a consequence, for all $f \in \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$ and for all $\varphi \in \mathcal{D}(\Omega)$, it holds

$${}^*\int_{{}^*\mathbb{R}^k} l'(f) {}^*\varphi dx = {}^*\int_{{}^*\mathbb{R}^k} \widehat{f} {}^*\varphi dx.$$

By S-continuity of ${}^*\varphi$, we have also

$${}^*\int_{{}^*\mathbb{R}^k} \widehat{f} {}^*\varphi dx \approx \langle f, {}^*\varphi \rangle.$$

If we let $f = l(g)$ for some $g \in L^p(\Omega)$, from Theorem 2.19 we have

$$\langle l(g), {}^*\varphi \rangle \approx \langle g, \varphi \rangle_{\mathcal{D}(\Omega)} = \int_{\mathbb{R}^k} g \varphi dx.$$

By putting together the previous equalities, we conclude that equation 11 holds. \square

We conjecture that for $p = 2$ and under the hypothesis that Ω has finite Lebesgue measure then by an appropriate choice of the embedding l defined in Theorem 2.19, we could have $\|(l' \circ l)(f) - {}^*f\|_2 \approx 0$, as in the original embedding by Robinson and Bernstein.

3.2. Grid functions as parametrized measures. It is well known that weak limits of L^p functions behave badly with respect to composition with a nonlinear function [2, 30, 53, 59]. Consider for instance a bounded sequence $\{u_n\}_{n \in \mathbb{N}}$ of $L^\infty(\Omega)$ functions: by the Banach–Alaoglu theorem, there is a subsequence of $\{u_n\}_{n \in \mathbb{N}}$ that has a weak- \star limit $u_\infty \in L^\infty(\Omega)$. Now let $f \in C_b^0(\mathbb{R})$: the sequence $\{f(u_n)\}_{n \in \mathbb{N}}$ is still bounded in $L^\infty(\Omega)$, so it has a weak- \star limit f_∞ . However, in general $f_\infty \neq f(u_\infty)$. To overcome this difficulty, the weak- \star limit of the sequence $\{u_n\}_{n \in \mathbb{N}}$ can be represented by a Young measure, i.e. a measurable function $\nu : \Omega \rightarrow \mathbb{M}^{\mathbb{P}}(\mathbb{R})$ such that for

all $f \in C_b^0(\mathbb{R})$ the weak- \star limit of $\{f(u_n)\}_{n \in \mathbb{N}}$ is the function defined a.e. by $\bar{f}(x) = \int_{\mathbb{R}} f d\nu_x$, in the sense that the equality

$$(12) \quad \lim_{n \rightarrow \infty} \int_{\Omega} f(u_n(x))g(x)dx = \int_{\Omega} \left(\int_{\mathbb{R}} f d\nu_x \right) g(x)dx = \int_{\Omega} \bar{f}(x)g(x)dx$$

holds for all $g \in L^1(\Omega)$.

Example 3.10. *The following example is discussed in detail in [59]. Consider the Rademacher functions $u_n(x) = u_0(n^2x)$, with $u_0(x) = \chi_{[0,1/2)}(x) - \chi_{[1/2,1)}(x)$ extended periodically over \mathbb{R} . It can be calculated that the Young measure ν associated to the sequence $\{u_n\}_{n \in \mathbb{N}}$ is constant and that*

$$\nu_x = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}$$

for almost every $x \in \Omega$, i.e. that for all $f \in C_b^0(\mathbb{R})$ and for all $g \in L^1(\Omega)$,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} f(u_n(x))g(x)dx = \left(\frac{1}{2}f(1) + \frac{1}{2}f(-1) \right) \int_{\mathbb{R}} g(x)dx.$$

In the setting of grid functions, instead of bounded sequences of L^∞ functions, we have grid functions with finite ${}^*L^\infty$ norm. These functions can be used to represent weak- \star limits of L^∞ functions.

Example 3.11. *The function $u(n\varepsilon) = (-1)^n$ can be thought as a representative for the weak- \star limit of the Rademacher functions: in fact, for all $f \in C_b^0(\mathbb{R})$ and for all $\varphi \in C_c^0(\Omega)$,*

$${}^\circ \langle {}^*f(u), {}^*\varphi \rangle = \left(\frac{1}{2}f(1) + \frac{1}{2}f(-1) \right) \int_{\mathbb{R}} \varphi(x)dx.$$

Since $C_c^0(\Omega)$ is dense in $L^1(\Omega)$, this is sufficient to conclude that the above formula holds for all $\varphi \in L^1(\Omega)$.

We will now make precise the connection between grid functions and Young measure by showing that every grid function that has finite ${}^*L^\infty$ norm corresponds to a Young measure. The proof of the following theorem relies on a result by Cutland [20].

Theorem 3.12. *For every $u \in {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$ with $\|u\|_\infty \in {}^*\mathbb{R}_{fin}$, there exists a Young measure $\nu^u : \Omega \rightarrow \mathbb{M}^{\mathbb{P}}(\mathbb{R})$ such that, for all $f \in C_b^0(\mathbb{R})$ and for all $\varphi \in C_c^0(\Omega)$,*

$$(13) \quad {}^\circ \langle {}^*f(u), {}^*\varphi \rangle = \int_{\Omega} \left(\int_{\mathbb{R}} f d\nu_x^u \right) \varphi(x)dx.$$

Proof. Since $\|u\|_\infty \in {}^*\mathbb{R}_{fin}$, there exists $n \in \mathbb{R}$ such that $|u(x)| < n$. We can identify u with a function $\tilde{u} : \hat{\Omega} \rightarrow {}^*\mathbb{M}^{\mathbb{P}}([-n, n])$ defined by $\tilde{u}(x) = \delta_{\hat{u}(x)}$.

Notice that for all $f \in C_b^0(\mathbb{R})$ and for all $\varphi \in C_c^0(\Omega)$ it holds

$$\begin{aligned}
 \langle {}^*f(u), {}^*\varphi \rangle &\approx {}^*\int_{\hat{\Omega}} {}^*f(\hat{u}(x)) {}^*\varphi(x) dx \\
 (14) \qquad &= {}^*\int_{\hat{\Omega}} \left({}^*\int_{[-n,n]} {}^*f d\tilde{u}(x) \right) {}^*\varphi(x) dx.
 \end{aligned}$$

We define an internal measure μ over ${}^*\Omega \times {}^*[-n, n]$ by posing

$$\mu(A \times B) = {}^*\int_A \tilde{u}_x(B) dx$$

for all Borel $A \subseteq \Omega$ and for all Borel $B \subseteq {}^*[-n, n]$. Let L_μ be the Loeb measure obtained from μ (for the properties of the Loeb measure, we refer for instance to [1, 37, 38]). We can define a standard measure μ_s over $\Omega \times [-n, n]$ by posing

$$\mu_s(A \times B) = L_\mu(\{x \in {}^*\Omega \times {}^*[-n, n] : {}^\circ x \in A \times B\}).$$

Since μ_s satisfies $\mu_s(A \times [-n, n]) = \mu_L(A)$ for all Borel $A \subseteq \Omega$, by Rohlin's Disintegration Theorem the measure μ_s can be disintegrated as

$$\mu_s(A \times B) = \int_A \nu_x^u(B) dx,$$

with $\nu^u : \Omega \rightarrow \mathbb{M}^{\mathbb{P}}([-n, n])$. By Lemma 2.6 of [20], ν^u satisfies

$${}^\circ \left({}^*\int_{\Omega} \left({}^*\int_{[-n,n]} {}^*f d\tilde{u}(x) \right) {}^*\varphi(x) dx \right) = \int_{\Omega} \left(\int_{[-n,n]} f d\nu_x^u \right) \varphi(x) dx.$$

for all $f \in C_b^0(\mathbb{R})$ and for all $\varphi \in C_c^0(\Omega)$. Thanks to equality 14, we deduce that ν^u satisfies 13. We can extend ν_x^u to all of $\mathbb{M}^{\mathbb{P}}(\mathbb{R})$ by defining $\nu_x^u(A) = \nu_x^u(A \cap [-n, n])$ for all Borel sets $A \subseteq \mathbb{R}$ and for all $x \in \Omega$, thus obtaining a Young measure that satisfies equation 13. \square

In [3, 59], it is shown that Young measures describe weak- \star limits of bounded sequences of L^∞ functions. We will now show that grid functions with finite L^∞ norm can be similarly used to represent weak- \star limits of L^∞ functions in the setting of grid functions. This is a consequence of a more general property of the correspondence between grid functions and Young measures: if $u \in {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$ satisfies $\|u\|_\infty \in {}^*\mathbb{R}_{fin}$ and ν^u is the Young measure associated to u in the sense of Theorem 3.12, then $[u]$ corresponds to the barycentre of ν^u .

Theorem 3.13. *Let $u \in {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$ with $\|u\|_\infty \in {}^*\mathbb{R}_{fin}$, and let ν^u be the Young measure that satisfies equality 13. Then $[u] \in L^\infty(\Omega)$ and the following equality holds for a.e. $x \in \Omega$:*

$$(15) \qquad [u](x) = \int_{\mathbb{R}} \tau d\nu_x^u.$$

Moreover,

- (1) if $\{u_n\}_{n \in \mathbb{N}}$ is a sequence of L^∞ functions that converges weakly- \star to ν^u in the sense of equation 12, then $u_n \xrightarrow{\star} [u]$ in L^∞ ;
- (2) if ν^u is Dirac, then ν_x^u is the Dirac measure centred at $[u](x)$ for a.e. $x \in \Omega$.

Proof. Define a function f_ν by posing $f_\nu(x) = \int_{\mathbb{R}} \tau d\nu_x^u$ for all $x \in \Omega$. Since $|f_\nu(x)| \leq {}^\circ\|u\|_\infty$ for a. e. $x \in \Omega$ and since $\|u\|_\infty \in {}^*\mathbb{R}_{fin}$, $f_\nu \in L^\infty(\Omega)$. By Theorem 3.12, for all $\varphi \in C_c^0(\Omega)$ we have the following equalities:

$$\int_{\Omega} f_\nu(x) \varphi(x) dx = \int_{\Omega} \int_{\mathbb{R}} \tau d\nu_x \varphi(x) dx = {}^\circ\langle u, {}^*\varphi \rangle = \int_{\Omega} [u] \varphi dx.$$

Since $C_c^0(\Omega)$ is dense in $L^1(\Omega)$, we deduce that $f_\nu = [u]$ in $L^\infty(\Omega)$, as we wanted.

We will now prove (1). By hypothesis, from equation 12 and from equation 13, it holds

$$\lim_{n \rightarrow \infty} \int_{\Omega} f(u_n(x)) \varphi(x) dx = \int_{\Omega} \left(\int_{\mathbb{R}} f d\nu_x \right) \varphi(x) dx = {}^\circ\langle {}^*f(u), {}^*\varphi \rangle$$

for all $\varphi \in C_c^0(\Omega)$. As a consequence, by considering a function $f \in C_b^0(\mathbb{R})$ with $f(x) = 1$ for all x satisfying $|x| \leq {}^\circ\|u\|_\infty$, we obtain that the weak- \star limit of the sequence $\{u_n\}_{n \in \mathbb{N}}$ is equal to $[u]$.

Assertion (2) is a consequence of equality 15. \square

If the sequence $\{u_n\}_{n \in \mathbb{N}}$ is not bounded in L^∞ , but it is bounded in $L^p(\Omega)$ for some $1 \leq p < \infty$, then it can be proved that there exists a parametrized measure $\nu : \Omega \rightarrow \mathbb{M}(\mathbb{R})$ such that for all $f \in C_b^0(\mathbb{R})$ the weak- \star limit of the sequence $\{f(u_n)\}_{n \in \mathbb{N}}$ is the function defined a.e. by $\bar{f}(x) = \int_{\mathbb{R}} f d\nu_x$ (for an in-depth discussion of this result, we refer to [3]). Notice that ν takes values in $\mathbb{M}(\mathbb{R})$ instead of $\mathbb{M}^{\mathbb{P}}(\mathbb{R})$, since the sequence $\{u_n\}_{n \in \mathbb{N}}$ could diverge in a subset of Ω with positive measure.

The grid function counterpart of this result is that for any $u \in {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$ there exists a function $\nu^u : \Omega \rightarrow \mathbb{M}(\mathbb{R})$ that satisfies equation 13, even if $\|u\|_\infty \notin {}^*\mathbb{R}_{fin}$. If $\|u\|_\infty \notin {}^*\mathbb{R}_{fin}$, ν_x^u might not be a probability measure, but it still satisfies the inequalities $0 \leq \nu_x^u(\mathbb{R}) \leq 1$ for all $x \in \Omega$. In particular, the difference between $\nu_x^u(\mathbb{R})$ and 1 is due to u being unlimited at some non-negligible fraction of $\mu(x) \cap \mathbb{X}^k$.

Theorem 3.14. *For every $u \in {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$, there exists a parametrized measure $\nu^u : \Omega \rightarrow \mathbb{M}(\mathbb{R})$ such that, for all $f \in C_b^0(\mathbb{R})$ and for all $\varphi \in C_c^0(\Omega)$, equality 13 holds. Moreover, for all $x \in \Omega$ and for all Borel $A \subseteq \mathbb{R}$, $0 \leq \nu_x^u(A) \leq 1$.*

Proof. Let $u \in {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$, and for all $n \in \mathbb{N}$ define

$$u_n(x) = \begin{cases} u(x) & \text{if } |u(x)| \leq n, \\ n & \text{if } u(x) > n, \\ -n & \text{if } u(x) < -n. \end{cases}$$

Since for all $n \in \mathbb{N}$ it holds $\|u_n\|_\infty \leq n \in {}^*\mathbb{R}_{fin}$, by Theorem 3.12 there exists a Young measure ν^n that satisfies

$$(16) \quad {}^\circ \langle {}^*f(u_n), {}^*\varphi \rangle = \int_{\Omega} \left(\int_{\mathbb{R}} f d\nu_x^n \right) \varphi(x) dx.$$

for all $f \in C_b^0(\mathbb{R})$ and for all $\varphi \in C_c^0(\Omega)$.

Recall that a sequence of parametrized measures $\{\mu^n\}_{n \in \mathbb{N}}$ converges weakly- \star to a parametrized measure μ if for all $f \in C_b^0(\mathbb{R})$, the sequence $\{f_n\}_{n \in \mathbb{N}}$ of L^∞ functions defined by

$$f_n(x) = \int_{\mathbb{R}} f d\mu_x^n$$

converges weakly- \star to a function $f \in L^\infty(\Omega)$ defined by

$$f(x) = \int_{\mathbb{R}} f d\mu_x.$$

Define ν^u as the parametrized measure satisfying $\nu^n \xrightarrow{\star} \nu^u$ for some subsequence (not relabelled) of $\{\nu^n\}_{n \in \mathbb{N}}$. The existence of such a weak- \star limit can be obtained as a consequence of the Banach-Alaouglu theorem (for further details about the weak- \star limit of measures, we refer to [30]). We claim that ν^u satisfies equality 13 and that for all $x \in \Omega$, $0 \leq \nu_x^u(\mathbb{R}) \leq 1$.

Let $f \in C_b^0(\mathbb{R})$. Since $\lim_{|x| \rightarrow \infty} f(x) = 0$, there is an increasing sequence of natural numbers $\{n_i\}_{i \in \mathbb{N}}$ such that if $|x| \geq n_i$, then $|f(x)| \leq 1/i$. As a consequence of this inequality, for all $i \in \mathbb{N}$ and for all $\varphi \in C_c^0(\Omega)$ it holds

$$| \langle {}^*f(u_{n_i}), {}^*\varphi \rangle - \langle {}^*f(u), {}^*\varphi \rangle | \leq 2/i \|{}^*\varphi\|_1.$$

Taking into account equation 16, from the previous inequality we obtain

$$\left| \int_{\Omega} \left(\int_{\mathbb{R}} f d\nu_x^{n_i} \right) \varphi(x) dx - {}^\circ \langle {}^*f(u), {}^*\varphi \rangle \right| \leq 2/i \|\varphi\|_1.$$

As a consequence, we deduce that

$$\lim_{i \rightarrow \infty} \int_{\Omega} \left(\int_{\mathbb{R}} f d\nu_x^{n_i} \right) \varphi(x) dx = {}^\circ \langle {}^*f(u), {}^*\varphi \rangle.$$

This is sufficient to entail that $\nu^n \xrightarrow{\star} \nu^u$ and that ν^u satisfies equality 13.

The inequality $0 \leq \nu_x^u(A) \leq 1$ for all Borel $A \subseteq \mathbb{R}$ is a consequence of the lower semicontinuity of the weak- \star limit of measures (see for instance theorem 3 of [30]). \square

Notice that, as a consequence of Theorem 3.14, we deduce that the hypothesis $\|u\|_\infty \in {}^*\mathbb{R}_{fin}$ in Theorem 3.12 can be relaxed. In particular, if v differs from u at some null set, then u and v induce the same parametrized measure, even if $u \not\equiv v$.

Corollary 3.15. *Let L_N be the Loeb measure obtained from the measure $\mu_N(A) = |A|/N^k$ for all internal $A \subseteq \mathbb{X}^k$. If for $u, v \in {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$ it holds*

$L_N(\{x \in \Omega_{\mathbb{X}} : u(x) \not\approx v(x)\}) = 0$, then $\nu^u = \nu^v$. If $\|u - v\|_p \approx 0$, then $\nu^u = \nu^v$.

Proof. If $L_N(\{x \in \Omega_{\mathbb{X}} : u(x) \not\approx v(x)\}) = 0$, then also

$$L_N(\{x \in \Omega_{\mathbb{X}} : *f(u(x)) \not\approx *f(v(x))\}) = 0$$

for all $f \in C_b^0(\mathbb{R})$. This and the hypothesis $f \in C_b^0(\mathbb{R})$ are sufficient to deduce $\langle *f(u), *\varphi \rangle \approx \langle *f(v), *\varphi \rangle$ for all $\varphi \in C_c^0(\Omega)$ that, thanks to equation 13, is equivalent to the equality $\nu^u = \nu^v$.

The hypothesis $\|u - v\|_p \approx 0$ implies $L_N(\{x \in \Omega_{\mathbb{X}} : u(x) \not\approx v(x)\}) = 0$, so the equality between ν^u and ν^v is a consequence of the previous part of the proof. \square

The above corollary can be seen as the grid function counterpart of Corollary 3.14 of [59], that shows how Young measure ignore concentration phenomena. We find it useful to discuss this behaviour with an example, that also highlights how a grid function can describe simultaneously very different properties of a sequence of L^p functions.

Example 3.16. *The following example is discussed from the standard viewpoint in [59]. Consider the sequence $\{u_n\}_{n \in \mathbb{N}}$ defined by $u_n(x) = n\chi_{[1-1/n, 1]}$. Notice that $\|u_n\|_\infty = n$, so that the sequence is not bounded in $L^\infty(\mathbb{R})$. For all $f \in C_b^0(\mathbb{R})$ and for all $\varphi \in C_c^0(\mathbb{R})$, it holds*

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} f(u_n) \varphi dx = f(0) \int_{\mathbb{R}} \varphi dx$$

so that the sequence $\{u_n\}_{n \in \mathbb{N}}$ converges weakly- \star to the constant Young measure $\nu_x = \delta_0$ for all $x \in \mathbb{R}$.

The sequence $\{u_n\}_{n \in \mathbb{N}}$ satisfies the L^1 uniform bound $\|u_n\|_1 = 1$ for all $n \in \mathbb{N}$. Since for all $\varphi \in \mathcal{D}(\mathbb{R})$ it holds

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} u_n \varphi dx = \lim_{n \rightarrow \infty} n \int_{[1-1/n, 1]} \varphi dx = \varphi(1)$$

the sequence $\{u_n\}_{n \in \mathbb{N}}$ converges in the sense of distributions to δ_1 , the Dirac distribution centred at 1. Indeed, it can be proved that the sequence $\{u_n\}_{n \in \mathbb{N}}$ converges weakly- \star to δ_1 in the space $\mathbb{M}(\mathbb{R})$ of Radon measures.

In the setting of grid functions, a representative for the limit of the sequence $\{u_n\}_{n \in \mathbb{N}}$ is given by $u_N = N\chi_1$. For all $f \in C_b^0(\mathbb{R})$ and for all $\varphi \in C_c^0(\mathbb{R})$, it holds

$$\langle *f(u_N), *\varphi \rangle = \varepsilon \sum_{x \in \mathbb{X}, x \neq 1} f(0) * \varphi(x) + \varepsilon * f(N) \varphi(1).$$

Since $f \in C_b^0(\mathbb{R})$, $*f(N) \approx 0$ and, by Lemma 1.9, we deduce

$${}^\circ \langle *f(u_N), *\varphi \rangle = f(0) \int_{\mathbb{R}} \varphi(x) dx.$$

From the above equality and from equation 13, we deduce that the Young measure associated to u_N is the constant Young measure $\nu_x = \delta_0$ for all

$x \in \mathbb{R}$. Notice that the same result could have been deduced from Corollary 3.15 by noticing that, since $L_N(\{x \in \Omega_{\mathbb{X}} : u_N(x) \not\approx 0\}) = 0$, the Young measure associated to u_N is the same as the Young measure associated to the constant function $c(x) = 0$ for all $x \in {}^*\mathbb{R}$.

As for the distribution corresponding to $[u_N]$, since for all $\varphi \in \mathcal{D}_{\mathbb{X}}(\mathbb{X})$ it holds $\langle N\chi_1, \varphi \rangle = \varphi(1)$, we deduce that $[u_N] = \delta_1$. In particular, the grid function u_N coherently describes the behaviour of the limit of the sequence $\{u_n\}_{n \in \mathbb{N}}$ both in the sense of Young measures and in the sense of distributions.

In the previous example we have considered a grid function u with $\|u\|_1 \in {}^*\mathbb{R}_{fin}$, and we verified that the parametrized measure associated to u was indeed a Young measure. This result holds under the more general hypothesis that $\|u\|_p \in {}^*\mathbb{R}_{fin}$.

Proposition 3.17. *If $\|u\|_p \in {}^*\mathbb{R}_{fin}$, then ν_x^u is a probability measure for a.e. $x \in \Omega$.*

Proof. If for some $x \in \Omega$ it holds $\nu_x^u(\mathbb{R}) < 1$, then there exists $y \in \Omega_{\mathbb{X}}$, $y \approx x$ such that $u(y) \notin {}^*\mathbb{R}_{fin}$. The hypothesis $\|u\|_p \in {}^*\mathbb{R}_{fin}$ implies $L_N(\{y \in \Omega_{\mathbb{X}} : u(y) \notin {}^*\mathbb{R}_{fin}\}) = 0$: this is sufficient to conclude that $\mu_L(\{x \in \Omega : \nu_x^u(\mathbb{R}) < 1\}) = 0$, as desired. \square

We will conclude the discussion of the relations between grid functions and parametrized measures by determining the parametrized measure associated to a periodic grid function with an infinitesimal period. This is the grid function counterpart of the formula for the Young measure associated to the limit of a sequence of periodic functions (see Example 3.5 of [2]). We will prove this result for $k = 1$, as the generalization to an arbitrary dimension is mostly a matter of notation.

Proposition 3.18. *If $u \in {}^*\mathbb{R}^{\mathbb{X}}$ is periodic of period $M\varepsilon \approx 0$, then the parametrized measure ν associated to u is constant, and*

$$\int_{\mathbb{R}} f d\nu_x = {}^\circ \left(\frac{1}{M} \sum_{i=0}^{M-1} {}^*f(u(i\varepsilon)) \right)$$

for all $x \in \Omega$ and for all $f \in C_b^0(\mathbb{R})$.

Proof. Without loss of generality, let $M \in {}^*\mathbb{N}$ and let u be periodic over $[0, (M-1)\varepsilon] \cap \mathbb{X}$, with $M\varepsilon \approx 0$.

Let $f \in C_b^0(\mathbb{R})$. At first, we will prove that $\frac{1}{M} \sum_{i=0}^{M-1} {}^*f(u(i\varepsilon))$ is finite: in fact,

$$(17) \quad \inf_{x \in {}^*\mathbb{R}} {}^*f(x) \leq \frac{1}{M} \sum_{i=0}^{M-1} {}^*f(u(i\varepsilon)) \leq \sup_{x \in {}^*\mathbb{R}} {}^*f(x)$$

and by the boundedness of f , we deduce that $\frac{1}{M} \sum_{i=0}^{M-1} {}^*f(u(i\varepsilon))$ is finite.

Let now $\varphi \in S^0(\mathbb{X})$ with $\text{supp } \varphi \subset [a, b]$, $a, b \in {}^*\mathbb{R}_{fin}$. Then there exists $h, k \in {}^*\mathbb{N}$ satisfying $a \approx Mh\varepsilon$ and $b \approx Mk\varepsilon$. We have the equalities

$$\begin{aligned}
 \langle {}^*f(u), \varphi \rangle &\approx \varepsilon \sum_{x \in [Mh\varepsilon, Mk\varepsilon]_{\mathbb{X}}} f(u(x)) \varphi(x) \\
 &= \varepsilon \sum_{j=h}^k \left(\sum_{i=0}^{M-1} {}^*f(u(i\varepsilon)) \varphi(jM\varepsilon + i\varepsilon) \right) \\
 &= \varepsilon \sum_{j=h}^k \left(\left(\sum_{i=0}^{M-1} {}^*f(u(i\varepsilon)) \right) (\varphi(jM\varepsilon) + e(j)) \right) \\
 (18) \quad &= \left(\frac{1}{M} \sum_{i=0}^{M-1} {}^*f(u(i\varepsilon)) \right) \left(M\varepsilon \sum_{j=h}^k (\varphi(jM\varepsilon) + e(j)) \right).
 \end{aligned}$$

Let

$$e = \max_{0 \leq i \leq M, k \leq j \leq h} \{|\varphi(jM\varepsilon) - \varphi(jM\varepsilon + i\varepsilon)|\}.$$

Since $\varphi \in S^0(\mathbb{X})$ and $\text{supp } \varphi \subset {}^*\mathbb{R}_{fin}$, $e \approx 0$ and, as a consequence, $|e(j)| \leq e \approx 0$. We deduce

$$\left| M\varepsilon \sum_{j=k}^h e(j) \right| \leq M\varepsilon(k-h)e \approx (b-a)e \approx 0$$

and, by equation 17,

$$(19) \quad \left(\frac{1}{M} \sum_{i=0}^{M-1} {}^*f(u(i\varepsilon)) \right) \left(M\varepsilon \sum_{j=h}^k e(j) \right) \approx 0.$$

Since $M\varepsilon \approx 0$,

$$(20) \quad M\varepsilon \sum_{j=h}^k (\varphi(jM\varepsilon)) \approx \int_{\circ a}^{\circ b} \circ \varphi(x) dx.$$

Putting together equalities 18, 19 and 20, we conclude

$$\circ \langle {}^*f(u), \varphi(x) \rangle = \circ \left(\frac{1}{M} \sum_{i=0}^{M-1} {}^*f(u(i\varepsilon)) \right) \int_{\circ a}^{\circ b} \circ \varphi(x) dx$$

as we wanted. \square

4. THE GRID FUNCTION FORMULATION OF PARTIAL DIFFERENTIAL EQUATIONS

In this section, we will give some results that allow to coherently formulate stationary and time-dependent PDEs in the sense of grid functions in a way that, if the solutions to the grid function formulation are regular enough, they induce standard solutions to the original problem. Moreover, the existence of regular solutions to the original problem is equivalent to the

existence of regular solutions to the grid function formulation. If the original problem does not have solutions in the sense of distributions, then we regard the solution to the grid function formulation as a generalized solution. It turns out that, for some nonlinear problems, the generalized solution in the sense of grid functions is related with some notions of measure-valued solutions.

4.1. The grid function formulation of linear PDEs. A linear PDE can be written in the most general form as

$$(21) \quad L(u) = f,$$

with $f \in \mathcal{D}'(\Omega)$, where $L : \mathcal{D}'(\Omega) \rightarrow \mathcal{D}'(\Omega)$ is linear, and where the equality is meant in the sense of distributions, i.e.

$$\langle L(u), \varphi \rangle_{\mathcal{D}(\Omega)} = \langle f, \varphi \rangle_{\mathcal{D}(\Omega)}$$

for all $\varphi \in \mathcal{D}_{\mathbb{X}}(\Omega)$. We would like to turn problem 21 in a problem in the sense of grid functions, i.e.

$$(22) \quad L_{\mathbb{X}}(u) = f_{\mathbb{X}},$$

with $f_{\mathbb{X}} \in \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$ and where $L_{\mathbb{X}} : \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}}) \rightarrow \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$ is ${}^*\mathbb{R}$ -linear. Moreover, we would like to determine sufficient conditions that ensure equivalence between problem 22 and problem 21, in the sense that 21 has a solution if and only if 22 has a solution.

Such a coherent formulation of linear PDEs relies upon the existence of ${}^*\mathbb{R}$ -linear extensions of linear functionals over the space of distributions. Recall that every linear functional $L : \mathcal{D}'(\Omega) \rightarrow \mathcal{D}'(\Omega)$ induces an adjoint $M : \mathcal{D}_{\mathbb{X}}(\Omega) \rightarrow \mathcal{D}_{\mathbb{X}}(\Omega)$ that satisfies

$$\langle L(T), \varphi \rangle_{\mathcal{D}(\Omega)} = \langle T, M(\varphi) \rangle_{\mathcal{D}(\Omega)}$$

for all $T \in \mathcal{D}'(\Omega)$ and for all $\varphi \in \mathcal{D}(\Omega)$. If we find a ${}^*\mathbb{R}$ -linear extension of M in the sense of grid functions, by taking the adjoint we are able to define a ${}^*\mathbb{R}$ -linear extension of L .

Lemma 4.1. *For every linear $L : \mathcal{D}(\Omega) \rightarrow \mathcal{D}(\Omega)$ there is a ${}^*\mathbb{R}$ -linear $L_{\mathbb{X}} : {}^*\mathbb{R}^{\Omega_{\mathbb{X}}} \rightarrow {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$ such that $L_{\mathbb{X}}({}^*\varphi) = {}^*(L(\varphi))|_{\Omega_{\mathbb{X}}}$ for all $\varphi \in \mathcal{D}(\Omega)$.*

Proof. For $\varphi \in \mathcal{D}'(\Omega)$ define

$$U(\varphi) = \{L_{\mathbb{X}} : {}^*\mathbb{R}^{\Omega_{\mathbb{X}}} \rightarrow {}^*\mathbb{R}^{\Omega_{\mathbb{X}}} \text{ such that } L_{\mathbb{X}} \text{ is } {}^*\mathbb{R}\text{-linear and } L_{\mathbb{X}}({}^*\varphi) = {}^*L(\varphi)|_{\Omega_{\mathbb{X}}}\}$$

and let $U = \{U(\varphi) : \varphi \in \mathcal{D}(\Omega)\}$. If we prove that U has the finite intersection property, then, by saturation, $\bigcap U \neq \emptyset$, and any $L_{\mathbb{X}} \in \bigcap U$ is a ${}^*\mathbb{R}$ -linear function that satisfies $L_{\mathbb{X}}({}^*\varphi) = {}^*(L(\varphi))_{\mathbb{X}}$ for all $\varphi \in \mathcal{D}(\Omega)$.

We will prove that, if $\varphi_1, \dots, \varphi_n \in \mathcal{D}$, then $\bigcap_{i=1}^n U(\varphi_i) \neq \emptyset$ by induction over n . If $n = 1$, we need to show that $U(\varphi) \neq \emptyset$ for all $\varphi \in \mathcal{D}'(\Omega)$. If $\varphi = 0$, then the constant function $L_{\mathbb{X}}f = 0$ for all $f \in {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$ belongs to $U(\varphi)$. If

$\varphi \neq 0$, let $f = {}^*\varphi|_{\Omega_{\mathbb{X}}}$, $g = {}^*(L(\varphi))|_{\Omega_{\mathbb{X}}}$, and let $\{f, b_2, \dots, b_M\}$ be a ${}^*\mathbb{R}$ -basis of ${}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$. Define also

$$L_{\mathbb{X}} \left(a_1 f + \sum_{i=2}^M a_i b_i \right) = a_1 g.$$

By definition, $L_{\mathbb{X}}$ is ${}^*\mathbb{R}$ -linear and $L_{\mathbb{X}} \in U(\varphi)$.

We will now show that if $\bigcap_{i=1}^{n-1} U(\varphi_i) \neq \emptyset$ for any choice of $\varphi_1, \dots, \varphi_{n-1} \in \mathcal{D}(\Omega)$, then also $\bigcap_{i=1}^n U(\varphi_i) \neq \emptyset$ for any choice of $\varphi_1, \dots, \varphi_n \in \mathcal{D}(\Omega)$. If $\{\varphi_1, \dots, \varphi_n\}$ are linearly dependent, thanks to linearity of L , any $L_{\mathbb{X}} \in \bigcap_{i=1}^{n-1} U(\varphi_i)$ satisfies $L_{\mathbb{X}} \in \bigcap_{i=1}^n U(\varphi_i)$. If $\{\varphi_1, \dots, \varphi_n\}$ are linearly independent, let $f_n = ({}^*\varphi_n)|_{\Omega_{\mathbb{X}}}$, let $g_n = {}^*(L(\varphi_n))|_{\Omega_{\mathbb{X}}}$ and let $\{f_n, b_2, \dots, b_M\}$ be a ${}^*\mathbb{R}$ -basis of ${}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$. For any $L_{\mathbb{X}} \in \bigcap_{i=1}^{n-1} U(\varphi_i)$, define $\overline{L}_{\mathbb{X}} : {}^*\mathbb{R}^{\Omega_{\mathbb{X}}} \rightarrow {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$ by

$$\overline{L}_{\mathbb{X}} \left(a_1 f_n + \sum_{i=2}^M a_i b_i \right) = a_1 g_n + L_{\mathbb{X}} \left(\sum_{i=2}^M a_i b_i \right).$$

Then $\overline{L}_{\mathbb{X}} \in \bigcap_{i=1}^n U(\varphi_i)$. This concludes the proof. \square

Theorem 4.2. *For every linear $L : \mathcal{D}'(\Omega) \rightarrow \mathcal{D}'(\Omega)$ there is a ${}^*\mathbb{R}$ -linear $L_{\mathbb{X}} : {}^*\mathbb{R}^{\Omega_{\mathbb{X}}} \rightarrow {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$ such that ${}^\circ \langle L_{\mathbb{X}} f, {}^*\varphi \rangle = \langle L[f], \varphi \rangle_{\mathcal{D}(\Omega)}$ for all $\varphi \in \mathcal{D}(\Omega)$. Moreover, if $L_{\mathbb{X}}(\mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})) \subseteq \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$, the following diagram commutes:*

$$(23) \quad \begin{array}{ccc} \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}}) & \xrightarrow{L_{\mathbb{X}}} & \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}}) \\ \Phi \circ \pi \downarrow & & \downarrow \Phi \circ \pi \\ \mathcal{D}'(\Omega) & \xrightarrow{L} & \mathcal{D}'(\Omega). \end{array}$$

Proof. Let M be the adjoint of L , and let $M_{\mathbb{X}}$ be the ${}^*\mathbb{R}$ -linear operator coherent with M in the sense of Lemma 4.1. Define $\langle L_{\mathbb{X}}(f), \varphi \rangle = \langle f, M_{\mathbb{X}}(\varphi) \rangle$ for all $\varphi \in \mathcal{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$. From this definition, ${}^*\mathbb{R}$ -linearity of $L_{\mathbb{X}}$ can be deduced from the ${}^*\mathbb{R}$ -linearity of $M_{\mathbb{X}}$.

We will now prove that $L_{\mathbb{X}}$ satisfies ${}^\circ \langle L_{\mathbb{X}} f, {}^*\varphi \rangle = \langle L[f], \varphi \rangle_{\mathcal{D}(\Omega)}$ for all $\varphi \in \mathcal{D}(\Omega)$. For any $\varphi \in \mathcal{D}(\Omega)$, thanks to Lemma 4.1 we have the equalities

$$\langle L_{\mathbb{X}}(f), {}^*\varphi \rangle = \langle f, M_{\mathbb{X}}({}^*\varphi|_{\Omega_{\mathbb{X}}}) \rangle = \langle f, {}^*M(\varphi) \rangle \approx \langle [f], M(\varphi) \rangle_{\mathcal{D}(\Omega)} = \langle L[f], \varphi \rangle_{\mathcal{D}(\Omega)},$$

as we wanted.

If $L_{\mathbb{X}}(f) \in \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$, then for all $\varphi \in \mathcal{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$

$$\langle L_{\mathbb{X}}(f), \varphi \rangle = \langle L_{\mathbb{X}}(f), \varphi - {}^*({}^\circ \varphi) \rangle + \langle L_{\mathbb{X}}(f), {}^*({}^\circ \varphi) \rangle.$$

Since $\varphi - {}^*({}^\circ \varphi) \equiv 0$, by Lemma 2.7 the hypothesis $L_{\mathbb{X}}(f) \in \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$ allows to conclude that $\langle L_{\mathbb{X}}(f), \varphi - {}^*({}^\circ \varphi) \rangle \approx 0$, so that ${}^\circ \langle L_{\mathbb{X}}(f), \varphi \rangle = \langle L[f], {}^\circ \varphi \rangle_{\mathcal{D}(\Omega)}$. As a consequence, the hypothesis $L_{\mathbb{X}}(\mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})) \subseteq \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$ is sufficient to entail that diagram 23 commutes. \square

From the previous Theorem, we obtain some sufficient conditions that ensure the equivalence between the linear problem 21 in the sense of distributions and the linear problem 22 in the sense of grid functions.

Theorem 4.3. *Let $L : \mathcal{D}'(\Omega) \rightarrow \mathcal{D}'(\Omega)$ be linear, and let $L_{\mathbb{X}} : {}^*\mathbb{R}^{\Omega_{\mathbb{X}}} \rightarrow {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$ any function such that diagram 23 commutes. Let also $f \in \mathcal{D}'(\Omega)$. Then problem 21 has a solution if and only if problem 22 has a solution $u \in \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$ for some $f_{\mathbb{X}}$ satisfying $[f_{\mathbb{X}}] = f$.*

Proof. By Theorem 4.2, if problem 22 has a solution u , then $[u]$ satisfies problem 21.

The other implication is a consequence of Theorem 4.2 and of surjectivity of Φ : suppose that 21 has a solution v . The commutativity of diagram 23 ensures that for any $u \in \Phi^{-1}(v)$ it holds $[L_{\mathbb{X}}(u)] = f$, hence for $f_{\mathbb{X}} = L_{\mathbb{X}}(u)$ problem 22 has a solution. \square

Thanks to this equivalence result, any linear PDE that admits an extension $L_{\mathbb{X}}$ such that diagram 23 commutes can be studied in the setting of grid functions with the techniques from linear algebra.

As an example of the grid function formulation of a linear PDE, we find it useful to discuss the Dirichlet problem.

Definition 4.4. *Let $\Omega \subset \mathbb{R}^k$ be open and bounded, $h \in \mathbb{N}$, $a_{\alpha,\beta} \in C^\infty(\Omega)$, and let*

$$L(v) = \sum_{0 \leq |\alpha|, |\beta| \leq h} (-1)^{|\alpha|} D^\alpha (a_{\alpha,\beta} D^\beta v).$$

The Dirichlet problem is the problem of finding v satisfying

$$(24) \quad \begin{cases} L(v) = f \text{ in } \Omega \\ D^\alpha u = 0 \text{ for } |\alpha| \leq h-1 \text{ in } \partial\Omega. \end{cases}$$

If $f \in C_b(\Omega)$, then v is a classical solution of the Dirichlet problem if

$$(25) \quad v \in C_b^{2h}(\Omega) \cap C_b^{2h-1}(\overline{\Omega}) \text{ and } L(v) = f.$$

If $f \in L^2(\Omega)$, then v is a strong solution of the Dirichlet problem if

$$v \in H^{2h}(\Omega) \cap H_0^h(\overline{\Omega}) \text{ and } L(v) = f \text{ a.e.}$$

If $f \in H^{-h}(\Omega)$, then v is a weak solution of the Dirichlet problem if

$$(26) \quad v \in H_0^h(\Omega) \text{ and } \sum_{0 \leq |\alpha|, |\beta| \leq h} \int a_{\alpha,\beta} D^\beta v D^\alpha w = f(w) \text{ for all } w \in H_0^h(\Omega).$$

Definition 4.5. *A grid function formulation of the Dirichlet problem 24 is the following: let*

$$L_{\mathbb{X}}(u) = \sum_{0 \leq |\alpha|, |\beta| \leq h} (-1)^{|\alpha|} \mathbb{D}^\alpha ({}^*a_{\alpha,\beta} \mathbb{D}^\beta u).$$

The Dirichlet problem is the problem of finding $u \in {}^\mathbb{R}^{\Omega_{\mathbb{X}}}$ satisfying*

$$(27) \quad \begin{cases} L_{\mathbb{X}}(u) = P({}^*f) \text{ in } \Omega_{\mathbb{X}} \\ \mathbb{D}^\alpha u = 0 \text{ in } \partial_{\mathbb{X}}^\alpha \Omega_{\mathbb{X}} \text{ for } |\alpha| \leq s-1. \end{cases}$$

Notice that equation 27 is satisfied in the sense of grid functions, i.e. point-wise, while equation 24 assumes the different meanings shown in Definition 4.4.

A priori, a solution u of problem 22 induces a solution $[u]$ of problem 21 in the sense of distributions. However, if $[u]$ is more regular, it is a solution to 21 in a stronger sense.

Theorem 4.6. *Let u be a solution of problem 27. Then*

- (1) *if $f \in C_b(\Omega)$ and $[u] \in C_b^{2h}(\Omega) \cap C_b^{2h-1}(\overline{\Omega})$, then $[u]$ is a classical solution of the Dirichlet problem;*
- (2) *if $f \in L^2(\Omega)$ and $[u] \in H^{2h}(\Omega) \cap H_0^h(\Omega)$, then $[u]$ is a strong solution of the Dirichlet problem;*
- (3) *if $f \in H^{-h}(\Omega)$ and $[u] \in H_0^h(\Omega)$, then $[u]$ is a weak solution of the Dirichlet problem, i.e. $[u]$ satisfies 26.*

Proof. A solution u of problem 27 satisfies the equality

$$\begin{aligned} \langle P(*f), \phi \rangle &= \sum_{0 \leq |\alpha|, |\beta| \leq h} (-1)^{|\alpha|} \langle \mathbb{D}^\alpha (*a_{\alpha, \beta} \mathbb{D}^\beta u), \varphi \rangle \\ &= \sum_{0 \leq |\alpha|, |\beta| \leq h} \langle *a_{\alpha, \beta} \mathbb{D}^\beta u, \mathbb{D}^\alpha \varphi \rangle. \end{aligned}$$

for all $\varphi \in \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$.

We will now prove (1). If $f \in C_b(\Omega)$, then by Lemma 3.5, $[P(*f)] = f$. By Theorem 2.15, $[\mathbb{D}^\beta u] = D^\beta[u]$, $[*a_{\alpha, \beta} \mathbb{D}^\beta u] = a_{\alpha, \beta} D^\beta[u]$, so that

$$[(-1)^{|\alpha|} \mathbb{D}^\alpha (*a_{\alpha, \beta} \mathbb{D}^\beta u)] = (-1)^{|\alpha|} D^\alpha (a_{\alpha, \beta} D^\beta[u]).$$

We deduce that $[u]$ satisfies equation 25 in the classical sense, as desired.

The proof of parts (2) and (3) is similar to that of part (1). The only difference is that it relies on Lemma 3.2 instead of Lemma 3.5. \square

Remark 4.7. *While Theorem 4.2 and Theorem 4.3 do not explicitly determine an extension $L_{\mathbb{X}}$ for a given linear PDE, they determine a sufficient condition for problem 22 to be a coherent representation of problem 21 in the sense of grid function. In the practice, an explicit extension $L_{\mathbb{X}}$ of a linear $L : \mathcal{D}'(\Omega) \rightarrow \mathcal{D}'(\Omega)$ can be determined from L by taking into account that*

- *thanks to Theorem 2.15, derivatives can be replaced by finite difference operators;*
- *shifts can be represented in accord to Corollary 2.14;*
- *if $a \in C^\infty(\Omega)$, then $[*af] = a[f]$ for all $f \in \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$, since for all $\varphi \in \mathcal{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$, $*a\varphi \in \mathcal{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$, and we have the equalities*

$${}^\circ \langle *af, \varphi \rangle = {}^\circ \langle f, *a\varphi \rangle = \langle [f], a\varphi \rangle_{\mathcal{D}(\Omega)} = \langle a[f], \varphi \rangle_{\mathcal{D}(\Omega)}.$$

Similarly, we have not established a canonical representative $f_{\mathbb{X}}$ for f . However, observe that for all $g \in {}^\mathbb{R}^{\Omega_{\mathbb{X}}}$ and for all $x \in \Omega_{\mathbb{X}}$ it holds*

$$g(x) = \sum_{y \in \Omega_{\mathbb{X}}} g(y) N^k \chi_y(x)$$

Moreover, $\chi_y(x) = \chi_0(x - y)$, so that once a solution u_0 for the problem $L_{\mathbb{X}}u = N^k\chi_0$ is determined, a solution for $L_{\mathbb{X}}u = g$ can be determined from the above equality by posing

$$(28) \quad u_g(x) = \sum_{y \in \Omega_{\mathbb{X}}} g(y)u_0(x - y).$$

In fact, by linearity of $L_{\mathbb{X}}$ we have that, for all $x \in \Omega_{\mathbb{X}}$,

$$\begin{aligned} L_{\mathbb{X}}(u_g(x)) &= L_{\mathbb{X}}\left(\sum_{y \in \Omega_{\mathbb{X}}} g(y)u_0(x - y)\right) \\ &= \sum_{y \in \Omega_{\mathbb{X}}} g(y)L_{\mathbb{X}}(u_0(x - y)) \\ &= \sum_{y \in \Omega_{\mathbb{X}}} g(y)N^k\chi_0(x - y) \\ &= g(x). \end{aligned}$$

In particular, u_0 plays the role of a fundamental solution for problem 22, while equality 28 can be interpreted as the discrete convolution between g and u_0 . As a consequence, the study of a linear problem 22 can be carried out by determining the solutions to the problem $L_{\mathbb{X}}(u) = N^k\chi_0$.

4.2. The grid function formulation of nonlinear PDEs. A nonlinear PDE can be written in the most general form as

$$F(u) = f,$$

usually with $u \in V \subseteq L^2(\Omega)$ and $F : V \rightarrow W \subseteq L^2(\Omega)$. As in the linear case, the grid function formulation of nonlinear problems is based upon the possibility to coherently extend every continuous $F : L^2(\Omega) \rightarrow L^2(\Omega)$ to all of ${}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$. Since the proofs of the following theorems are based upon Lemma 3.7, we will impose the additional hypothesis that the Lebesgue measure of Ω is finite. Notice that, in contrast to what happened for Theorem 4.2, in the proof of Theorem 4.8, we will be able to explicitly determine a particular extension $F_{\mathbb{X}}$ for a given continuous $F : L^2(\Omega) \rightarrow L^2(\Omega)$.

Theorem 4.8. *Let $\mu_L(\Omega) < +\infty$ and let $F : L^2(\Omega) \rightarrow L^2(\Omega)$ be continuous. Then there is a function $F_{\mathbb{X}} : {}^*\mathbb{R}^{\Omega_{\mathbb{X}}} \rightarrow {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$ that satisfies*

- (1) *whenever $u, v \in {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$ are nearstandard in $L^2(\Omega)$, $\|u - v\|_2 \approx 0$ implies $\|F_{\mathbb{X}}(u) - F_{\mathbb{X}}(v)\|_2 \approx 0$;*
- (2) *for all $f \in L^2(\Omega)$, $[F_{\mathbb{X}}(P(*f))] = F(f)$.*

Proof. We will show that the function defined by $F_{\mathbb{X}}(u) = P(*F(\hat{u}))$ for all $u \in {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$ satisfies the thesis. By continuity of F , whenever u and v are nearstandard in $L^2(\Omega)$ we have

$$\|u - v\|_2 \approx 0 \text{ implies } \|*F(u) - *F(v)\|_2 \approx 0,$$

and, by Lemma 3.7,

$$\|*F(u) - *F(v)\|_2 \approx 0 \text{ implies } \|F_{\mathbb{X}}(u) - F_{\mathbb{X}}(v)\|_2 \approx 0,$$

hence (1) is proved.

We will now prove that $[F_{\mathbb{X}}(P(*f))] = F(f)$. By Lemma 3.7, we have $\|*f - P(*f)\|_2 \approx 0$ and, by continuity of $*F$, $\|*F(*f) - *F(P(*f))\|_2 \approx 0$. From Lemma 3.6 we have $[F_{\mathbb{X}}(*f)] = [P(*F(*f))] = F(f)$, as desired. \square

Remark 4.9. *In the same spirit, if $F : V \rightarrow W$ is continuous and the space of grid functions can be continuously embedded in $*V$ and $*W$, then one can prove similar theorems by varying condition (1) in order to properly represent the topologies on the domain and the range of F . For instance, if $F : H^1(\Omega) \rightarrow L^2(\Omega)$, then (1) would be replaced by*

$$\|u - v\|_{H^1} \approx 0 \text{ implies } \|F_{\mathbb{X}}(u) - F_{\mathbb{X}}(v)\|_2 \approx 0,$$

where $\|u - v\|_{H^1}$ is defined in the expected way as

$$\|u - v\|_{H^1} = \|u - v\|_2 + \|\nabla_{\mathbb{X}}(u - v)\|_2.$$

Condition (1) of Theorem 4.8 is a continuity requirement for $F_{\mathbb{X}}$, and condition (2) implies coherence of $F_{\mathbb{X}}$ with the original function F , so that theorem 4.8 ensures that for all continuous $F : L^2 \rightarrow L^2$ there is a function $F_{\mathbb{X}} : *R^{\Omega_{\mathbb{X}}} \rightarrow *R^{\Omega_{\mathbb{X}}}$ which is continuous and coherent with F . This result allows to formulate nonlinear PDEs in the setting of grid functions.

Theorem 4.10. *Let $\mu_L(\Omega) < +\infty$, let $F : L^2(\Omega) \rightarrow L^2(\Omega)$ and let $F_{\mathbb{X}} : *R^{\Omega_{\mathbb{X}}} \rightarrow *R^{\Omega_{\mathbb{X}}}$ satisfy conditions (1) and (2) of Theorem 4.8. Let also $f \in L^2(\Omega)$. Then the problem of finding $v \in L^2(\Omega)$ satisfying*

$$(29) \quad F(v) = f$$

*has a solution if and only if there exists a solution $u \in *R^{\Omega_{\mathbb{X}}}$, u nearstandard in $L^2(\Omega)$, that satisfy*

$$(30) \quad F_{\mathbb{X}}(u) = f_{\mathbb{X}}$$

*for some $f_{\mathbb{X}} \in *R^{\Omega_{\mathbb{X}}}$ with $[f_{\mathbb{X}}] = f$, and in particular for $f_{\mathbb{X}} = P(*f)$.*

Proof. Suppose that 30 with $f_{\mathbb{X}} = P(*f)$ has a solution u . Since $[P(*f)] = f$ by Corollary 3.6, u satisfies the equality $[F_{\mathbb{X}}(u)] = f$ in the sense of distributions. At this point, if u is nearstandard in $L^2(\Omega)$, by Lemma 3.7 we have $\|*[u] - u\|_2 \approx 0$, so that $[u] \in L^2(\Omega)$, and condition (2) of Theorem 4.8 ensures that $[F_{\mathbb{X}}(u)] = F([u])$, so that $[u]$ is a solution of 29.

For the other implication, suppose that v is a solution to 29. Then, by condition (2) of Theorem 4.8, $[F_{\mathbb{X}}(P(*v))] = F(v) = f$, so that problem 30 has a solution. \square

If u is a solution to 30 but it is not nearstandard in $L^2(\Omega)$, i.e. if $\|*[u] - u\|_2 \not\approx 0$, $[F_{\mathbb{X}}(u)]$ needs not be equal to $F([u])$. In fact, if $[u] \in L^2(\Omega)$ and $\|*[u] - u\|_2 \not\approx 0$, we have argued in Section 3.1 that we expect u to feature either strong oscillations or concentrations. Due to these irregularities,

we have no reasons to expect that $[F_{\mathbb{X}}(u)](x)$, that represents the mean of the values assumed by $F_{\mathbb{X}}(u)$ at points infinitely close to x , is related to $F([u])(x)$, that represents the function F applied to the mean of the values assumed by u at points infinitely close to x . However, as we have seen in Section 3.12, if $\|u\|_{\infty} \in {}^*\mathbb{R}_{fin}$, then u can be interpreted as a Young measure ν^u . If the composition $F(\nu^u)$ is defined in the sense of Equation 13, then ν^u satisfies

$$\int_{\Omega} \int_{\mathbb{R}} F(\tau) d\nu^u(x) \varphi(x) dx = {}^{\circ}\langle F_{\mathbb{X}}(u), \varphi \rangle = {}^{\circ}\langle P(*f), \varphi \rangle = \int_{\Omega} f \varphi dx$$

for all $\phi \in \mathcal{D}'(\Omega)$, and can be regarded as a Young measure solution to Equation 29. In particular, since Young measures describe weak- \star limits of sequences of L^{∞} functions, the relation between $F(\nu^u)$ and problem 29 is the following: there exists a family of regularized problems

$$F_{\eta}(u) = f_{\eta}$$

and a family $\{u_{\eta}\}_{\eta>0}$ of $L^2(\Omega) \cap L^{\infty}(\Omega)$ solutions of these problems such that ν^u represents the weak- \star limit of a subsequence of $\{u_{\eta}\}_{\eta>0}$, and $F(\nu^u)$ is the corresponding weak limit of the sequence $\{F(u_{\eta})\}_{\eta>0}$.

In the case that $\|u\|_{\infty}$ is infinite or that $[u] \notin L^2(\Omega)$, we consider u as a generalized solution of problem 29 in the sense of grid functions. Moreover, we expect u to capture both the oscillations and the concentrations we would expect from a sequence of solutions of some family of regularized problems of 29. A more in-depth example of this behaviour is discussed in the grid function formulation of a class of ill-posed PDEs in [12].

Remark 4.11. Notice that if $F_{\mathbb{X}}$ satisfies the stronger continuity hypothesis

$$(31) \quad u \equiv v \text{ implies } F_{\mathbb{X}}(u) \equiv F_{\mathbb{X}}(v),$$

then $F_{\mathbb{X}}$ has a standard part \tilde{F} defined by

$$\tilde{F}(g) = [F_{\mathbb{X}}(P(*g))]$$

for any $g \in L^2(\Omega)$. Moreover, from Lemma 3.7 and from Theorem 4.8, we deduce that $\tilde{F} = F$. As a consequence, any grid function u that satisfies $F_{\mathbb{X}}(u) = P(*f)$ induces a solution to problem 29.

However, the continuity condition 31 holds only for very regular functions, and it fails for many of the functions that still satisfy the hypotheses of Theorem 4.8.

Remark 4.12. If the function F appearing in equation 29 can be expressed as

$$F = L \circ G,$$

where G is nonlinear and L is linear, the equivalence between the standard notions of solutions for the PDE 29 and one of its formulations in the sense of grid functions can be obtained by a suitable combination of the results of Theorem 4.3 and of Theorem 4.10.

4.3. Time dependent PDEs. Time dependent PDEs have been studied in the setting of nonstandard analysis by a variety of means. A possibility is to give a nonstandard representation of a given time dependent PDE by discretizing in time as well as in space, and by defining a standard solution to the original problem by the technique of stroboscopy. In [10], van den Berg showed how the stroboscopy technique can be extended to the study of a class of partial differential equations of the first and the second order by imposing additional regularity hypotheses on the time-step of the discretization. For an in-depth discussion on the stroboscopy technique and its applications to partial differential equations, we remand to [10, 47, 48].

A delicate point in the time discretization of PDEs is that the discrete time step cannot be chosen arbitrarily. In fact, it is often the case that the time-step of the discretization must be chosen in accord to some bounds that depend upon the specific problem. As an example, consider the nonstandard model for the heat equation discussed in [32], where the time-step is dependent upon the diameter of the grid and upon the diffusion coefficients. In general, if the discrete timeline \mathbb{T} is a deformation of the grid \mathbb{X} , then the finite difference in time does not generalize faithfully the partial difference in time, and Theorem 1.15 fails. However, it is possible to determine sufficient conditions over \mathbb{T} that imply the existence of $k \in \mathbb{N}$ such that Theorem 1.15 holds for derivatives up to order k . This study has been carried out in depth by van den Berg in [9].

In order to provide a general theory that is not dependent upon the specific problem, we have chosen to follow the idea of Capiński and Cutland in [13, 16] and subsequent works: we will not discretize in time, but instead we will work with functions defined on ${}^*\mathbb{R} \times \mathbb{X}^k$, where the first variable represents time, and the other k variables represent space. In particular, we want to describe the problem

$$(32) \quad u_t - Fu = f$$

with $u : \mathbb{R} \rightarrow V \subseteq L^2(\Omega)$, $F : V \rightarrow W \subseteq L^2(\Omega)$ by the nonstandard problem

$$(33) \quad u_t - F_{\mathbb{X}}u = f_{\mathbb{X}}$$

with $u : {}^*\mathbb{R} \rightarrow {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$, with $[f_{\mathbb{X}}] = f$, and where $F_{\mathbb{X}}$ is a suitable extension of F in the sense of Theorems 4.2 and 4.8. Notice that, by Theorem 2.15 and by Theorem 4.2, the grid function formulation of a time dependent PDE is formally a hyperfinite system of ordinary differential equations, and it can be solved by exploiting the standard theory of dynamical systems.

Once we have a grid function formulation for a time dependent PDE, we would like to study the relation between its solutions and the solutions to the original problem. As expected, if for a suitable choices of $F_{\mathbb{X}}$ problem 33 has a solution u and u is regular enough, then u induces a solution to problem 32.

Theorem 4.13. *Let $F_{\mathbb{X}}$ be coherent with F in the sense of Theorems 4.2 and 4.8. If 33 has a solution $u(t) \in {}^*C^1({}^*[0, T], {}^*\mathbb{R}^{\Omega_{\mathbb{X}}})$ that satisfies the*

continuity hypothesis

$$(34) \quad u(t) \equiv u(t') \text{ and } F(u(t)) \equiv F(u(t')) \text{ whenever } t \approx t',$$

then u induces two functions $[u], [F_{\mathbb{X}}(u)] \in C^0([0, T], \mathcal{D}'(\Omega))$ that satisfy

$$\int_{[0, T] \times \Omega} [u] \varphi_t + [F_{\mathbb{X}}(u)] \varphi d(t, x) + \int_{\Omega} [u(0, x)] \varphi(0, x) dx = - \int_{[0, T] \times \Omega} [f_{\mathbb{X}}] \varphi d(t, x)$$

for all $\varphi \in C^1([0, T], \mathcal{D}(\Omega))$ with $\varphi(T, x) = 0$.

Moreover, if F is linear or if $u(t)$ is nearstandard in the domain of F for all $t \in {}^*[0, T]$, then we can replace $[F_{\mathbb{X}}(u)]$ with $F([u])$ in the above equality.

Proof. By condition 34, the functions $t \mapsto [u(t)]$ and $t \mapsto [F(u(t))]$ are well-defined and continuous with respect to the weak- \star topology on $\mathcal{D}'(\Omega)$. Moreover, since $u \in {}^*C^1$,

$$\int_{{}^*[0, T]} \langle u_t, {}^*\varphi \rangle dt = - \int_{{}^*[0, T]} \langle u, {}^*\varphi_t \rangle dt - \langle {}^*u(0, x), {}^*\varphi(0, x) \rangle$$

for all $\varphi \in C^1([0, T], \mathcal{D}(\Omega))$ with $\varphi(T, x) = 0$. As a consequence, u satisfies the equality

$$\int_{[0, T]} \langle u, {}^*\varphi_t \rangle + \langle F_{\mathbb{X}}(u), {}^*\varphi \rangle dt + \langle {}^*u(0, x), {}^*\varphi(0, x) \rangle = - \int_{[0, T]} \langle f_{\mathbb{X}}, {}^*\varphi \rangle dt$$

for all $\varphi \in C^1([0, T], \mathcal{D}(\Omega))$ with $\varphi(T, x) = 0$, and this is equivalent to the first part of the thesis.

The second part of the thesis is a consequence of Theorem 4.2 and of Theorem 4.8. \square

If u does not satisfy 34 but $\|u(t)\|_{\infty}$ is finite and uniformly bounded in t , by the same argument of Theorem 3.12 u corresponds to a Young measure $\nu^u : [0, T] \times \Omega \rightarrow \mathbb{M}^{\mathbb{P}}(\mathbb{R})$. If the composition $F(\nu^u)$ is defined in the sense of Equation 13, then ν^u satisfies the equality

$$\begin{aligned} \int_{[0, T] \times \Omega} \int_{\mathbb{R}} \tau d\nu^u(t, x) \varphi_t + \int_{\mathbb{R}} F(\tau) d\nu^u(t, x) \varphi d(t, x) + \\ + \int_{\Omega} \int_{\mathbb{R}} \tau d\nu^u(0, x) \varphi(0, x) dx = \int_{[0, T] \times \Omega} [f_{\mathbb{X}}] \varphi d(t, x) \end{aligned}$$

for all $\varphi \in C^1([0, T], \mathcal{D}(\Omega))$ with $\varphi(T, x) = 0$.

If $\|u(t)\|_p$ is finite for some $1 \leq p < +\infty$, the sense in which $[u]$ is a solution to problem 32 has to be addressed on a case-by-case basis. In [12], we will discuss an example where $\|u(t)\|_1$ is finite and uniformly bounded, and $[u]$ can be interpreted as a Radon measure solution to problem 32.

5. SELECTED APPLICATIONS

In this section, we will discuss two classic problems: the first concerns the nonlinear theory of distributions, and the second is a minimization problem from the calculus of variations. The discussion of these examples is meant to show how grid functions can be applied to a variety of problems while retaining coherence with the various standard approaches.

5.1. The product HH' . The following example is discussed in the setting of Colombeau algebras in [18], and it can also be formalized in the framework of algebras of asymptotic functions [43].

Let H be the Heaviside function

$$H(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$$

and let H' be the derivative of the Heaviside function in the sense of distributions, i.e. the Dirac distribution centered at 0. It is well-known that the product HH' is not well-defined in the sense of distributions. However, this product arises quite naturally in the description of some physical phenomena. For instance, in the study of shock waves discussed in [18], it is convenient to treat H and H' as smooth functions and performing calculations such as

$$(35) \quad \int_{\mathbb{R}} (H^m - H^n) H' dx = \left[\frac{H^{m+1}}{m+1} \right]_{-\infty}^{+\infty} - \left[\frac{H^{n+1}}{n+1} \right]_{-\infty}^{+\infty} = \frac{1}{m+1} - \frac{1}{n+1}.$$

This calculation is not justified in the theory of distributions: on the one hand, $H^m = H^n$ for all $m, n \in \mathbb{N}$, so that we intuitively expect that the integral should equal 0; on the other hand, since the products $H^m H'$ and $H^n H'$ are not defined, the integrand is not well-defined.

We will now show how in the setting of grid functions one can rigorously formulate the integral 35 and compute the product HH' . Let $M \in {}^*\mathbb{N} \setminus \mathbb{N}$ satisfy $M\varepsilon \approx 0$, and consider the grid function $h \in \mathcal{D}'_{\mathbb{X}}(\mathbb{X})$ defined by

$$h(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x/(M\varepsilon) & \text{if } 0 < x < M\varepsilon \\ 1 & \text{if } x \geq M\varepsilon \end{cases}$$

The function $\mathbb{D}h$ is given by

$$\mathbb{D}h(x) = \begin{cases} 0 & \text{if } x \leq 0 \text{ and } x \geq M\varepsilon \\ 1/(M\varepsilon) & \text{if } 0 < x < M\varepsilon \end{cases}$$

In the next Lemma, we will prove that h is a representative of the Heaviside function for which the calculation 35 makes sense.

Lemma 5.1. *The function h has the following properties:*

- (1) $[h^m] = H$ and $[\mathbb{D}h^m] = \delta_0$ for all $m \in {}^*\mathbb{N}$;
- (2) $h^m \neq h^n$ whenever $m \neq n$;
- (3) $\langle h^m - h^n, \mathbb{D}h \rangle \approx \frac{1}{m+1} - \frac{1}{n+1}$.

Proof. (1). Let $\varphi \in \mathcal{D}_{\mathbb{X}}(\mathbb{X})$ and, without loss of generality, suppose that $\varphi(x) \geq 0$ for all $x \in \mathbb{X}$. Then for all $m \in {}^*\mathbb{N}$ we have the inequalities

$$\varepsilon \sum_{x \geq M\varepsilon} \varphi(x) \leq \langle h^m, \varphi \rangle \leq \varepsilon \sum_{x \geq 0} \varphi(x),$$

and, by taking the standard part of all the sides of the inequalities, we deduce

$$\int_0^{+\infty} {}^\circ \varphi(x) dx \leq {}^\circ \langle h^m, \varphi \rangle \leq \int_0^{+\infty} {}^\circ \varphi(x) dx.$$

This is sufficient to conclude that $[h^m] = H$ for all $m \in {}^*\mathbb{N}$. By Theorem 2.15, $[\mathbb{D}h^m] = H' = \delta_0$.

(2). Let $m \neq n$. Then,

$$(h^m - h^n)(x) = \begin{cases} 0 & \text{if } x \leq 0 \text{ and } x \geq M\varepsilon \\ (x/(M\varepsilon))^m - (x/(M\varepsilon))^n & \text{if } 0 < x < M\varepsilon. \end{cases}$$

In particular, $h^m - h^n \neq 0$, even if $[h^m] - [h^n] = 0$.

(3). By the previous point,

$$\langle h^m - h^n, \mathbb{D}h \rangle = \frac{1}{M} \sum_{j=1}^M (j/M)^m - (j/M)^n.$$

Since M is infinite,

$$\frac{1}{M} \sum_{j=1}^M (j/M)^m - (j/M)^n \approx \int_0^1 x^m - x^n dx = \frac{1}{m+1} - \frac{1}{n+1}.$$

□

Thanks to the lemma above, we can compute the equivalence class in $\mathcal{D}'(\mathbb{R})$ of the product hh' .

Corollary 5.2. $[h\mathbb{D}h] = \frac{1}{2}H'$.

Proof. For any $\varphi \in \mathcal{D}_{\mathbb{X}}(\mathbb{X})$, we have

$$\langle h\mathbb{D}h, \varphi \rangle = \frac{1}{M^2} \sum_{j=1}^M j\psi(j\varepsilon).$$

Let $\underline{m} = \min_{1 \leq j \leq M} \{\varphi(j\varepsilon)\}$ and $\overline{m} = \max_{1 \leq j \leq M} \{\varphi(j\varepsilon)\}$. We have the following inequalities:

$$\frac{\underline{m}}{M} \sum_{j=1}^M j/M \leq \frac{1}{M^2} \sum_{j=1}^M j\varphi(j\varepsilon) \leq \frac{\overline{m}}{M} \sum_{j=1}^M j/M.$$

Since M is infinite,

$$\frac{1}{M} \sum_{j=1}^M j/M \approx \int_0^1 x dx = \frac{1}{2},$$

so that

$$^\circ\left(\frac{m}{2}\right) \leq ^\circ\langle h\mathbb{D}h, \varphi \rangle \leq ^\circ\left(\frac{\overline{m}}{2}\right).$$

By S-continuity of φ , $\underline{m} \approx \overline{m} \approx \varphi(0)$, so that $^\circ\langle h\mathbb{D}h, \varphi \rangle = \frac{1}{2}^\circ\varphi(0)$ for all $\varphi \in \mathcal{D}_{\mathbb{X}}(\mathbb{X})$, which is equivalent to $[h\mathbb{D}h] = \frac{1}{2}H'$. \square

Notice that h is not the only function satisfying Lemma 5.1 and Corollary 5.2. In fact, we conjecture that Lemma 5.1 and Corollary 5.2 hold for a class of grid functions that satisfy some regularity conditions yet to be determined.

5.2. A variational problem without a minimum. We will now discuss a grid function formulation of a classic example of a variational problem without a minimum. For an in-depth analysis of the Young measure solutions to this problem we refer to [53], and for a discussion of a similar problem in the setting of ultrafunctions, we refer to [8]. The grid function formulation consists in a hyperfinite discretization, as in Cutland [21].

Consider the problem of minimizing the functional

$$(36) \quad J(u) = \int_0^1 \left(\int_0^x u(t) dt \right)^2 + (u(x)^2 - 1)^2 dx$$

with $u \in L^2([0, 1])$. Intuitively, a minimizer for J should have a small mean, but nevertheless it should assume values in the set $\{-1, +1\}$. Let us make precise this idea: define

$$u_0 = \chi_{[k, k+1/2)} - \chi_{[k+1/2, k+1)}, \quad k \in \mathbb{Z}$$

and let $u_n : [0, 1] \rightarrow \mathbb{R}$ be defined by $u_n(x) = u_0(nx)$. It can be verified that $\{u_n\}_{n \in \mathbb{N}}$ is a minimizing sequence for J , but J has no minimum. However, the sequence $\{u_n\}_{n \in \mathbb{N}}$ is uniformly bounded in $L^\infty([0, 1])$, hence it admits a weak* limit in the sense of Young measures. The limit is given by the constant Young measure

$$\nu_x = \frac{1}{2}(\delta_1 + \delta_{-1}).$$

We can now evaluate $J(\nu)$:

$$\int_0^x \nu(t) dt = \int_0^x \left(\int_{\mathbb{R}} \tau d\nu_x \right) dt = 0,$$

meaning that the barycentre of ν is 0, and

$$(\nu(x)^2 - 1)^2 = \int_{\mathbb{R}} (\tau^2 - 1)^2 d\nu_x = 0$$

since the support of ν is the set $\{-1, +1\}$. As a consequence, ν can be interpreted as a minimum of J in the sense of Young measures.

In the setting of grid functions, the functional 36 can be represented by

$$J_{\mathbb{X}}(u) = \varepsilon \sum_{n=0}^N \left[\left(\varepsilon \sum_{i=0}^n u(i\varepsilon) \right)^2 + (u(n\varepsilon)^2 - 1)^2 \right].$$

Observe that this representation is coherent with the informal description of J , and that the only difference between J and $J_{\mathbb{X}}$ is the replacement of the integrals with the hyperfinite sums. Let us now minimize $J_{\mathbb{X}}$ in the sense of grid functions. The minimizing sequence found in the classical case suggests us that a minimizer of $J_{\mathbb{X}}$ should assume values ± 1 , and that it should be piecewise constant in an interval of an infinitesimal length. For $M \in {}^*\mathbb{N}$, let $u_M = {}^*u_0(Mx)$. If $M < M' \leq N/2$, then

$$\varepsilon \sum_{i=0}^n u_M(i\varepsilon) > \varepsilon \sum_{i=0}^n u_{M'}(i\varepsilon).$$

We deduce that a minimizer for $J_{\mathbb{X}}$ is the grid function $u_{N/2}$, that is explicitly defined by $u_{N/2}(n\varepsilon) = (-1)^n$.

We will now show that this solution is coherent with the one obtained with the classic approach, i.e. that the Young measure associated to $u_{N/2}$ corresponds to $\frac{1}{2}(\delta_1 + \delta_{-1})$. Since $\|u_{N/2}\|_{\infty} = 1$, Theorem 3.12 guarantees the existence of a Young measure ν that corresponds to $u_{N/2}$. Moreover, by Proposition 3.18, ν is constant, and

$$\int_{\mathbb{R}} f d\nu_x = \frac{1}{2} \sum_{i=0}^1 f(u_{N/2}(i\varepsilon)) = \frac{1}{2}(f(1) + f(-1))$$

for all $f \in C_b^0(\mathbb{R})$. We deduce that the Young measure associated to $u_{N/2}$ is constant and equal to $\frac{1}{2}(\delta_1 + \delta_{-1})$, the minimizer of J in the sense of Young measures.

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